Estimating Nonlinear Heterogeneous Agents Models with Neural Networks

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Introduction

- HANK models have gained more and more prominence
  - Role of borrowing limits, social inequality, transmission of monetary policy
- Such models are hard to handle due to their elevated complexity
  - Heterogeneous agents facing idiosyncratic risks
  - Aggregate uncertainty and nonlinearities
- Forces to study tractable approximations and limits the empirical analysis
  - Loss of interesting features such as ZLB or stochastic volatility

⇒ New approach based on machine learning to estimate complex models

- Estimation of a HANK model in its nonlinear specification

Estimation with Neural Networks

- Neural networks (NN) are the fundamental building block of our approach
  - → Neural networks can tame curse-of-dimensionality and are very scalable
- Why is it challenging to estimate complex models?
  1. It is infeasible to solve such complex models sufficiently often

  → We exploit the scalability of NN to solve the model only ONCE
    - Treat parameters as pseudo state variables and adapt the NN training

  2. It is very costly to evaluate the likelihood with a Monte Carlo filter repeatedly

  → We develop a particle filter trained neural network approach
    - Training of additional NN to provide the outcome of the filter

Proofs of Concept

1. Neural network based solution vs. analytical one for a simple model
   - Solution based on our extended neural network with pseudo state variables
   - Laboratory model is a version of the linearized 3 equation NK model

   ⇒ Extended neural network coincides with true solution

2. Neural network based estimation vs. conventional one for a nonlinear model
   - Focus on Bayesian estimation with neural networks
   - Laboratory model is a RANK model with a zero lower bound

   ⇒ The estimation results are very similar

Estimating a Non-Linear HANK Model

- Nonlinear Heterogeneous Agent New Keynesian model as laboratory
  - Idiosyncratic income risk and borrowing limit for households
  - Several aggregate shocks, backward looking components and ZLB
  - Estimation includes 12 parameters
    - No restriction on the parameters that can be considered
  - Identification of parameters related to idiosyncratic risk is weak

$\Rightarrow$ Our estimation procedure can recover the true-data generating process

$\Rightarrow$ Capturing idiosyncratic and aggregate risk simultaneously is important

  - Interactions between nonlinearities, aggregate uncertainty and heterogeneity

Literature

- Neural Networks in Macroeconomic Modeling
  - Fernandez-Villaverde et al. (2020), Chen et al. (2021), Maliar, Maliar and Winant (2021), Azinovic et al. (2022)

  $\Rightarrow$ Neural network based likelihood estimation procedure

- HANK models, Aggregate Uncertainty and Nonlinearities
  - Reiter (2009), Ahn et al. (2018), Boppart et al.(2018), Auclert et al. (2021), Winberry (2021), Gorodnichenko et al. (2021), Fernandez-Villaverde et al.

  $\Rightarrow$ Strategy that exploits neural networks to estimate HANK models

- Estimation of HANK models
  - Auclert et al. (2020,2021), Bayer et al. (2019), Lee (2021)

  $\Rightarrow$ Estimation of nonlinear HANK model with individual and aggregate risk

Challenge for Estimation and Neural Networks

- Estimation of complex nonlinear models
  - E.g., HANK models with aggregate and individual nonlinearities
- Requires the repetition of two expensive steps again and again
  1. **Solve the model for a considered parameters combination**
  2. Evaluate the fit of the model with the data (with a particle filter)
- Seems to render estimation of complex models infeasible

⇒ A neural networks approach to overcome these issues

  - Proofs of concept and estimation of a HANK model

Class of Models

- Interested in solving DSGE models in its nonlinear specification
  - State variables $\mathbb{S}_t$, shocks ν_t and structural parameters Θ
- Dynamics of model can be summarized as (nonlinear) transition equation

$$\mathbb{S}_t = f\left(\mathbb{S}_{t-1}, \nu_t; \Theta\right),$$

where function f is generally unknown and needs to be obtained numerically

- Heterogeneity: Approximate distribution with finite number agents L More

Key trick: Pseudo State Variables

- Decisive step to solve these models is to obtain policy functions $\psi(\cdot)$:

$$\psi_t = \psi(\mathbb{S}_t | \Theta),$$

- Key trick: Solve the policy function over an entire parameter range
- Divide the parameters in two subsets

$$\Theta = \{\tilde{\Theta}, \bar{\Theta}\},$$

where $\tilde{\Theta}$ is the set of parameters to be estimated and $\bar{\Theta}$ is the set of parameters to be calibrated

- Treat the (subset of) parameters as pseudo state variables

$$\psi_t = \psi(\mathbb{S}_t, \tilde{\Theta} | \bar{\Theta}),$$

$\Rightarrow$ Policy functions depend now on the state variables and the parameters

Example: Linearized NK model

- Small off-the-shelf linearized three equation NK model with TFP shock
  - Features an analytical solution

$$\hat{X} = E_t\hat{X}_{t+1} - \sigma^{-1}\left(\phi_\Pi\hat{\Pi}_t + \phi_Y\hat{X}_t - E_t\hat{\Pi}_{t+1} - \hat{R}_t^F\right)$$
$$\hat{\Pi}_t = \kappa\hat{X}_t + \beta E_t\hat{\Pi}_{t+1}$$
$$\hat{R}_t^F = \rho_A\hat{R}_{t-1}^F + \sigma(\rho_A - 1)\omega\sigma_A\epsilon_t^A$$

where $\hat{X}_t$ is the output gap, $\hat{\Pi}$ is inflation, R_t^F is the risk free rate and ϵ_t^A is a TFP shock

Example: Solution to Linearized NK Model

- Solution to equation system depends on state variables and parameters

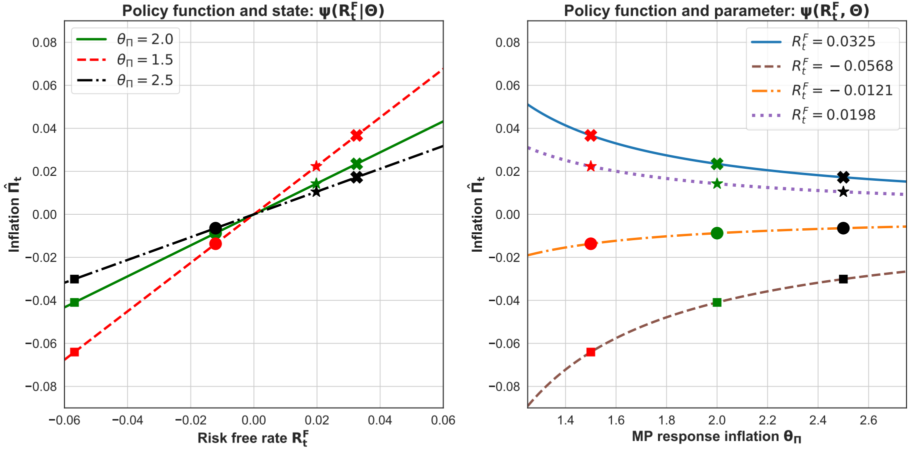
$$\begin{pmatrix} \hat{X}_t \\ \hat{\Pi}_t \end{pmatrix} = \psi \left(\underbrace{\hat{R}_t^F}_{\text{State } S_t}, \underbrace{\beta, \sigma, \eta, \phi, \theta_\Pi, \theta_Y, \rho_A, \sigma_A}_{\text{Parameters } \tilde{\Theta}} \right).$$

- The analytical solution is given as (method of undetermined coefficients)

$$\hat{X}_t = \frac{1 - \beta\rho_A}{(\sigma(1-\rho_A) + \theta_Y)(1-\beta\rho_A) + \kappa(\theta_\Pi - \rho_A)} \hat{R}_t^F,$$
$$\hat{\Pi}_t = \frac{\kappa}{(\sigma(1-\rho_A) + \theta_Y)(1-\beta\rho_A) + \kappa(\theta_\Pi - \rho_A)} \hat{R}_t^F.$$

Graphical Characterization: Policy Function

- Policy function over the parameter space $\theta_\Pi \in [1.25, 2.75]$



Challenge

- But, how can we solve such policy functions for models that feature jointly
  1. Pseudo state variables for estimation $\tilde{\Theta}$
  2. Nonlinear dynamics (e.g. zero lower bound, borrowing limits)
  3. Heterogeneous agents
- Most numerical techniques are not well suited due to curse of dimensionality

$\Rightarrow$ NN tame the curse of dimensionality and are universal approximators More

- Can handle high-dimensional input (e.g. many parameters, shocks, agents)
- Can resolve local features accurately (e.g. nonlinear features)
- Can capture irregularly shaped domain

Remarkable Features of Neural Networks

1. Universal approximation theorem (Hornik et al. 1989, Cybenko 1989)
   - Sufficient wide NN can approximate any finite-dimensional function with any desired non zero error

   $\Rightarrow$ NN can be used to solve macroeconomic models

2. Scalability and curse of dimensionality (Barron, 1993, Bach, 2017)
   - NN handle high dimensional problems much better than classical function approximators

   $\Rightarrow$ Scalability allows to handle models with a large number of states

$+$ Extraordinary efficiency of modern machine learning software and hardware

Extended Neural Network-Based Solution Method

- We use NN to solve the extended policy functions $\psi_{NN}(\mathbb{S}_t, \tilde{\Theta}|\bar{\Theta})$
  - Minimization of the Euler residual (loss function)
  - Training over thousands of iteration and number of economies (batch size B)
  - Training over parameter space and stochastic solution domain
- Adjust NN training to solve policy function over entire parameter space

$$\tilde{\Theta} = \left\{ \left[\tilde{\Theta}^{\underline{1}}, \tilde{\Theta}^{\overline{1}}\right], \left[\tilde{\Theta}^{\underline{2}}, \tilde{\Theta}^{\overline{2}}\right], \ldots, \left[\tilde{\Theta}^{\underline{P}}, \tilde{\Theta}^{\overline{P}}\right] \right\}$$

  - We draw new parameters for each economy in each iteration
  - Combined with a simulation step to adjust solution domain for each draw

⇒ Extended neural network provides solution over entire parameter space

Example: Linearized NK Model

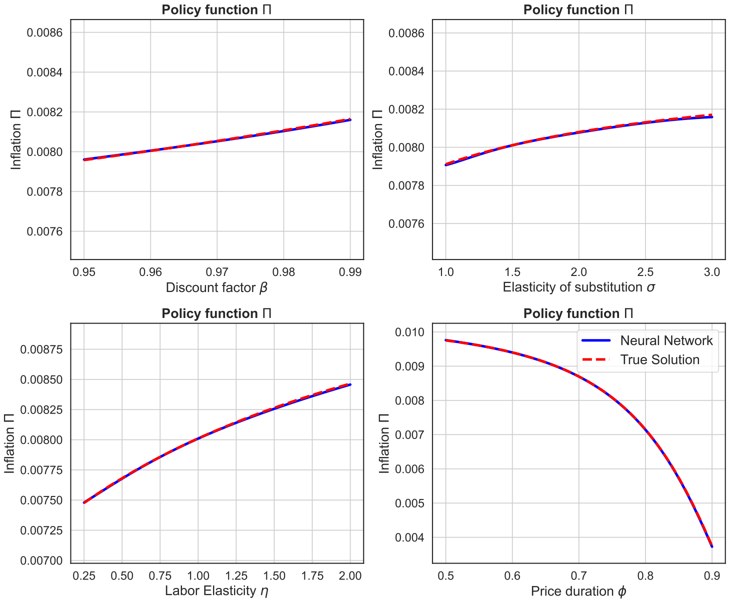
- We are interested in finding $\begin{pmatrix} \hat{X}_t \\ \hat{\Pi}_t \end{pmatrix} = \psi_{NN}\left(\hat{R}^F_t, \beta, \sigma, \eta, \phi, \theta_\Pi, \theta_Y, \rho_A, \sigma_A\right)$
- Minimization of the squared residual error (loss function)

$$
\begin{aligned}
err_1 =& \hat{X} - \left(E_t \hat{X}_{t+1} - \sigma^{-1}\left(\phi_\Pi \hat{\Pi}_t + \phi_Y \hat{X}_t - E_t \hat{\Pi}_{t+1} - \hat{R}^F_t\right)\right) \\
err_2 =& \hat{\Pi}_t - \left(\kappa \hat{X}_t + \beta E_t \hat{\Pi}_{t+1}\right)
\end{aligned}
$$

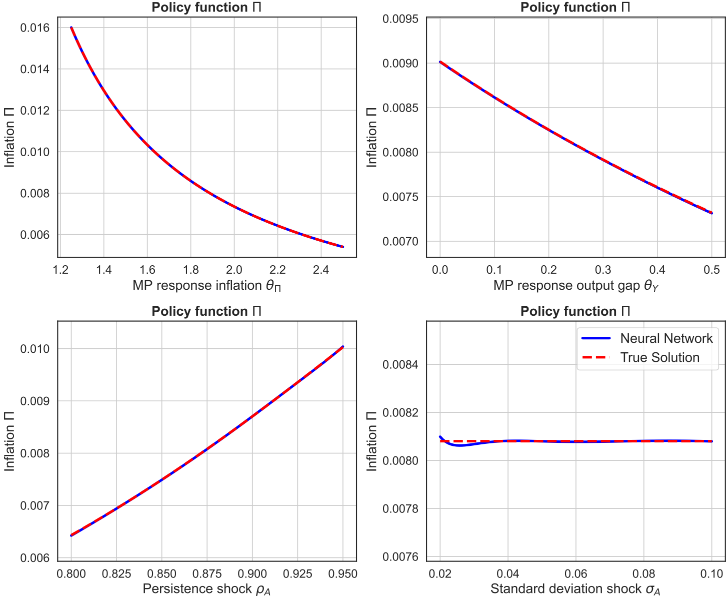
- Training NN over 100000 iterations and batch size of 500 economies
- Stochastic solution from simulating $\hat{R}^F_t = \rho_A \hat{R}^F_{t-1} + \sigma(\rho_A - 1)\omega\sigma_A\epsilon^A_t$
- We train the extended NN by drawing from the bounded parameter space

Parameters		LB	UB	Parameters		LB	UB
β	Discount factor	0.95	0.99	θ_Π	MP inflation response	1.25	2.5
σ	Relative risk aver.	1	3	θ_Y	MP output response	0.0	0.5
η	Inverse Frisch elas.	1	4	ρ_A	Persistence TFP shock	0.8	0.95
φ	Price duration	0.5	0.9	σ_A	Std. dev. TFP shock	0.02	0.1

Neural Network: Inflation over the Parameter Space



NN: Inflation over the Parameter Space (cont'd)



Challenge for Estimation and Neural Networks

- Estimation of complex nonlinear models
  - E.g., HANK models with aggregate and individual nonlinearities
- Requires the repetition of two expensive steps again and again
  1. ~~**Solve the model for a considered parameters combination**~~
  2. **Evaluate the fit of the model with the data (with a particle filter)**
- Seems to render estimation of complex models infeasible

⇒ A neural networks approach to overcome these issues
  - Proofs of concept and estimation of a HANK model

Neural Network Particle Filter

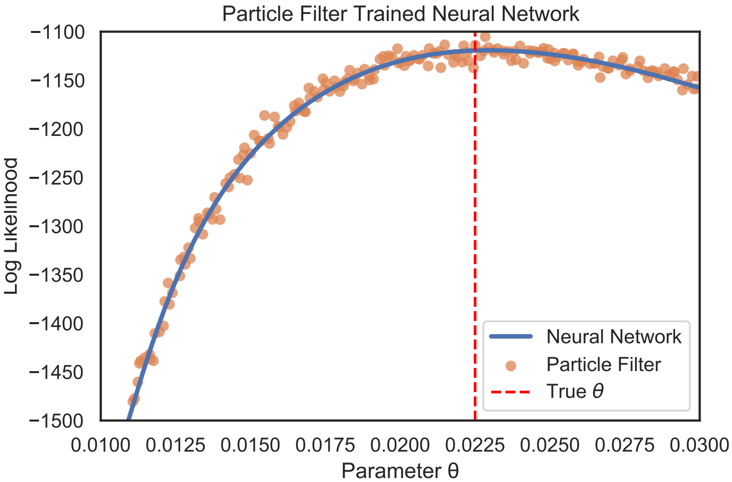
- We need to evaluate the fit of the model with the data More
  - Particle filter calculates the likelihood of the nonlinear model
  - Calculation is noisy and can be time consuming $\Rightarrow$ Bottleneck
- Goal: Additional neural network that gives directly output of particle filter
  - Create a dataset of parameter values and corresponding likelihoods
    - Run the particle filter for randomly drawn values from the parameter space
  - Train an additional neural network with this dataset

$\Rightarrow$ Particle filter trained neural network approach

- Surrogate model that provides mapping from parameters to likelihood

Graphical Characterization: NN based Particle Filter

- Particle filter trained neural network
  - Use particle filter to create data, which then can used to train neural network



Challenge for Estimation and Neural Networks

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  - **Proofs of concept and estimation of a HANK model**

Comparison to Conventional Estimation

- Estimation of a nonlinear model with neural networks
  - RANK model with zero lower bound in its fully nonlinear specification
  - True data-generating process to provide controlled environment
- Neural network based Bayesian estimation
  - Extended neural network, surrogate particle filter, RWMH algorithm
- Conventional methods follow Herbst and Schorfheide (2015)
  - Solve model with global methods, particle filter, RWMH algorithm

⇒ Estimation results are very similar and recover true data-generating process

More

Estimation of Nonlinear HANK with Neural Networks

- HANK with individual and aggregate nonlinearities
  - Households face idiosyncratic income risk s_t^i and a borrowing limit $\underline{B}$

$$E_0 \sum_{t=0}^{\infty} \beta^t \exp(\zeta_t^D) \left[\left(\frac{1}{1-\sigma} \right) (C_t^i - hC_{t-1})^{1-\sigma} - \chi \left(\frac{1}{1+\eta} \right) (H_t^i)^{1+\eta} \right]$$

$$\text{s.t. } C_t^i + B_t^i = W_t s_t^i H_t^i + \frac{R_{t-1}}{\Pi_t} B_{t-1}^i - T_t^i + Div_t^i$$

$$B_t \geq \underline{B}$$

  where idiosyncratic risk follows an AR(1) process: $s_t^i = \rho_s s_{t-1}^i + \sigma_s \epsilon_t^i$

  - Aggregate shocks: preference ζ^D, growth rate g_t and monetary policy mp_t
  - Consumption habit h and persistence in the monetary policy rule ρ_R
  - Monetary policy is constrained by the zero lower bound

$$R_t = \max \left[1, \left(R_{t-1}^N \right)^{\rho_R} \left(R \left(\frac{\Pi_t}{\Pi} \right)^{\theta_\Pi} \left(\frac{Y_t}{Z_t Y} \right)^{\theta_Y} \right)^{1-\rho_R} \exp(mp_t) \right]$$

Setup and Training of Neural Networks

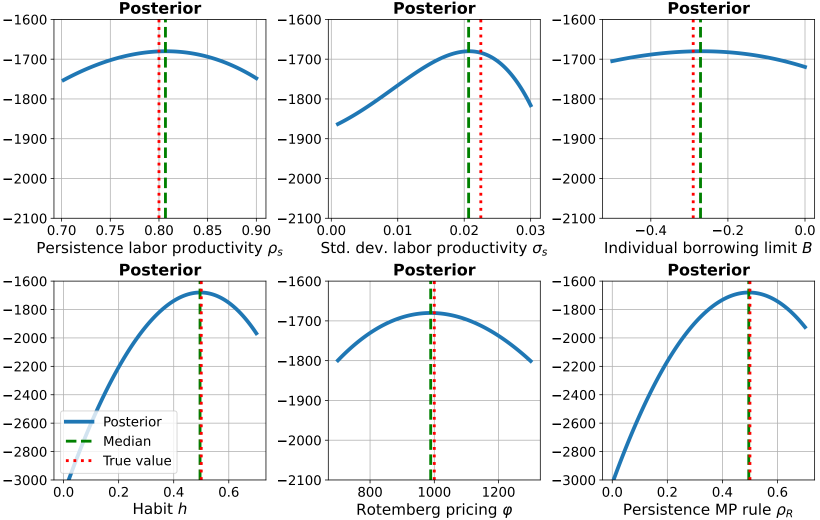
- We are interested in finding the policy functions over the parameter range
  - Aggregate PFs: Inflation and wage
  - Individual PFs: Labor choice and multiplier on borrowing constraint
- Model features 217 state variables
  - 200 individual, 5 aggregate and 12 pseudo (parameters) states
  - Approximation of continuum with 100 agents
- Training NN over 200000 iterations and batch size of 100 economies
  - Minimization of the squared residual error of 205 equations
- Estimation, likelihood and particle filter
  - Observation equation connects output growth, inflation and interest rate
  - NN based particle filter trained with 15000 likelihood points
  - Metropolis Hastings algorithm with 500000 draws

$\Rightarrow$ Bayesian estimation of HANK model in its nonlinear specification

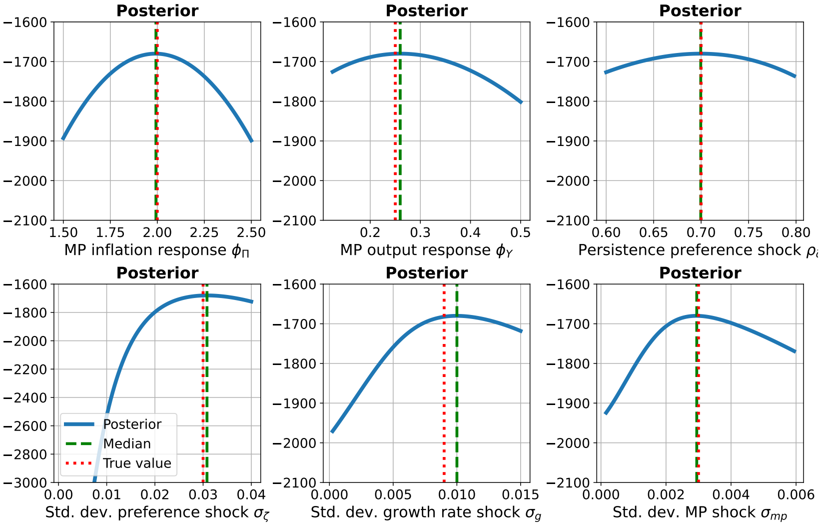
Estimation: Results and Priors

Estimation								
Par.	Prior					Neural Network		
	Type	Mean	Std	Lower Bound	Upper Bound	Posterior Median	5%	95%
Parameters related to idiosyncratic risk								
$\underline{B}$	Trc.N	-0.29	0.05	-0.5	0.0	-0.27	-0.36	-0.18
ρ_s	Trc.N	0.8	0.01	0.7	0.9	0.81	0.79	0.82
σ_s	Trc.N	2.25%	0.5%	0.01%	3.0%	2.07	1.87	2.26
Parameters related to aggregate risk								
h	Trc.N	0.5	0.01	0.0	0.7	0.50	0.48	0.51
φ	Trc.N	1000	25	700	1300	989	949	1028
ρ_r	Trc.N	0.5	0.01	0.0	0.7	0.50	0.48	0.51
θ_Π	Trc.N	2.0	0.025	1.5	2.5	2.00	1.95	2.03
θ_Y	Trc.N	0.25	0.025	0.125	0.5	0.26	0.23	0.29
ρ_ζ	Trc.N	0.7	0.025	0.6	0.8	0.70	0.68	0.72
σ_ζ	Trc.N	3.0%	0.25%	0.1%	4.0%	3.08%	2.92%	3.25%
σ_g	Trc.N	0.9%	0.1%	0.01%	1.5%	1.00%	0.92%	1.09%
σ_{mp}	Trc.N	0.3%	0.1%	0.01%	0.6%	0.29%	0.27%	0.32%

Posterior: Estimated Parameters and Idiosyncratic Risk



Posterior: Estimated Parameters and Aggregate Risk

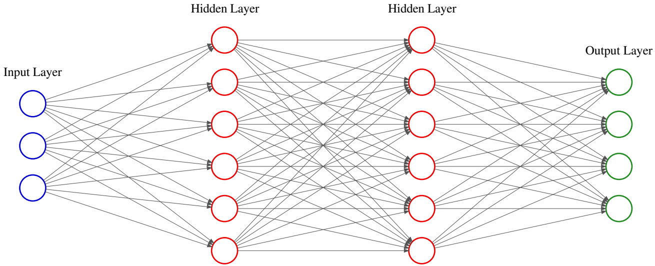


Conclusion

- Novel neural network based Bayesian estimation procedure
  - Extended neural network with pseudo state variables
  - Neural network based particle filter algorithm
  - Estimation of models with hundreds of state variables possible
- Estimation of a HANK with individual and aggregate nonlinearities
  - Two proofs of concept based on simpler models
- Techniques open up new exciting avenues for future research questions
  - Framework to think about monetary policy strategy and inequality

Primer on Neural Networks

- Deep learning uses deep neural networks (NN) as fundamental building block
- NN are mathematical function that maps some inputs into outputs
  - Composed of several layers with neutrons



- NN is trained with batch of data points to minimize a defined loss function

Back

Incorporation of Heterogeneity

- Heterogeneity usually assumes the existence of a continuum of agents
  - → Distribution of states and shocks is infinite

$$\int \mathbb{S}_t^i d\Omega \quad \text{and} \quad \int \nu_t^i d\Omega,$$

- We approximate the distribution with a finite number agents L

$$\left\{\mathbb{S}_t^i\right\}_{i=1}^L \quad \text{and} \quad \left\{\nu_t^i\right\}_{i=1}^L.$$

- The state variables and shock can be summarized as:

$$\mathbb{S}_t = \left\{\left\{\mathbb{S}_t^i\right\}_{i=1}^L, \mathbb{S}_t^A\right\} \quad \text{and} \quad \nu_t = \left\{\left\{\nu_t^i\right\}_{i=1}^L, \nu_t^A\right\},$$

- Individual and aggregate policy functions we adjust the policy functions

$$\psi_t^i = \psi^I(\mathbb{S}_t^i, \mathbb{S}_t|\bar{\Theta}) \quad \text{and} \quad \psi_t^A = \psi^A(\mathbb{S}_t|\bar{\Theta}).$$

Back

Particle Filter

- Observation equation connects the state variables with the observables $\mathbb{Y}_t$:

$$\mathbb{Y}_t = g(\mathbb{S}_t|\tilde{\Theta}) + u_t,$$

  where g is a function and u_t is a measurement error

- Particle filter determines the likelihood

$$\mathcal{L}\left(\mathbb{Y}_{1:T};\tilde{\Theta}\right) = \Omega^{PF}\left(\mathbb{Y}_{1:T};\tilde{\Theta}\right)$$

- Particle filter can be noisy and very time consuming for complex models
- Using a filter to calculate the likelihood is still a bottleneck Back

Nonlinear RANK Model with ZLB

- Off-the-shelf New Keynesian model
  - Shocks to households' preference to consumption
  - Price rigidities a la Rotemberg
  - Zero lower bound constraint on the nominal interest rate

$$R_t = \max\left[1, R\left(\frac{\Pi_t}{\Pi}\right)^{\theta_\Pi}\left(\frac{Y_t}{Z_t Y}\right)^{\theta_Y}\right]$$

- We are interested in solving and estimating it in its nonlinear specification

Back

Neural Network and Estimation

- Training NN over 100000 iterations and batch size of 200 economies
- We train the extended NN by drawing from the bounded parameter space
- Stochastic solution from simulating the model after each draw

- Observation equation with a sample size of 1000 periods

$$\begin{bmatrix} \text{Output Growth} \\ \text{Inflation} \\ \text{Interest Rate} \end{bmatrix} = \begin{bmatrix} 100 \ln \left(\frac{Y_t}{Y_{t-1}} \right) \\ 400 \ln \left(\Pi_t \right) \\ 400 \ln \left(R_t \right) \end{bmatrix} + u_t$$

- Estimation includes five structural parameters
- Priors are truncated normal densities
- 15000 data points to train neural network based particle filter

Estimation Results

Estimation							
Par.	Cal.	Neural Network			Conventional Approach		
	True Value	Posterior Median	5%	95%	Posterior Median	5%	95%
θ_Π	2.0	2.02	1.87	2.17	2.06	1.94	2.20
θ_Y	0.25	0.251	0.238	0.263	0.248	0.237	0.258
φ	1000	988.6	935.1	1036.7	973.7	911.2	1037.2
ρ_ζ	0.8	0.686	0.669	0.701	0.691	0.670	0.710
σ^ζ	0.02	0.020	0.020	0.021	0.020	0.019	0.020

- Neural network based estimation works very well
  - Posterior median is very close to the true value
- The bounds of neural network and conventional method are very similar
- Neural network method is much faster and much more scalable!

Bayesian Estimation with NN: Posterior