



BANK OF ENGLAND

NOTES ON THE BANK OF ENGLAND OPTION IMPLIED PROBABILITY DENSITY FUNCTIONS

Options are contracts used to insure against or speculate/take a view on uncertainty about the future prices of a wide range of financial assets and physical commodities. The prices at which options are traded contain information about the markets' uncertainty about the future prices of these 'underlying' assets. On certain assumptions the information in options prices can be expressed in terms of the probability that the price of the underlying asset will lie within particular ranges. The Macro Financial Analysis Division of the Bank of England estimates such implied 'probability density functions' (pdfs) for future values of a number of financial assets and commodities on a daily basis. These pdfs do not necessarily provide us with the actual probabilities of an asset price realising particular values in the future. Instead they can provide us with an idea of the probabilities that option market participants in aggregate attach to different outcomes. The methodology used to back-out an implied pdf is described in the Bank of England Quarterly Bulletin article by Clews, Panigirtzoglou and Proudman (2000). One assumption in the calculations is that market participants do not require compensation for risk (they are 'risk neutral'). The significance of this assumption is discussed in section 2 below. Some examples of how implied pdfs are used and interpreted by the Bank can be found in the Money & Asset Prices and Costs and Prices sections of the Bank's *Inflation Report*.¹ These background notes describe the relevant financial instruments, some terminology, and other issues of which the user should be aware.

The spreadsheets and zip files on the Bank of England internet site provide implied pdf related data for the FTSE 100 index and short sterling interest rates at a range of future dates/horizons:

- a) time series of summary statistics² that describe the location, range and shape of the implied pdf;
and,
- b) time series of percentiles of the cumulative probability distribution function (cdf).

¹ For example, see Bank of England *Inflation Report*, May 2002, Section 4.1, pg. 30.

² Exact definitions of the summary statistics are provided in section 3.

1 Types of instrument

Options and futures contracts

An option contract gives the holder the right, but not the obligation, to buy (in a call option) or sell (in a put option) a specified asset ('the underlying asset') at specified price (the 'exercise price' or 'strike price'). In a so-called European option the holder can choose to exercise this right at one specified date in the future (the 'expiry date' of the contract). In an American option the right to buy or sell can be exercised at or before the expiry date. The underlying assets are rarely actually exchanged. Instead, in the event that the right to buy/sell is exercised, the transaction is settled in cash (it is the difference between the strike price and the price of the underlying asset that changes hands). For a given asset, members of options exchanges trade contracts with a range of future expiry dates, and on a range of strike prices.

A call option, say, will only be worth exercising if, at the time, the price of the underlying asset is higher than the strike price. And the profit to be made from exercising the option will be greater the further the price of the underlying asset is above the strike price. The price that market participants will be willing to pay for the option will reflect their view of the chances of making such profits, ie the chances of the price of the underlying asset being at different points above the strike price. As already noted, options are generally traded at a series of different strike prices. For a given expiry date, the difference in price of options with different strike prices will reflect in part the market's view of the chances that the price of the underlying asset will end up between the strike prices. It is for this reason that we can back out the probabilities attached to different outcomes for a given asset price on a particular date in the future. Taken together, the probabilities across the possible future outcomes form a 'probability distribution function'.

Many options contracts are related to underlying assets which are themselves traded – for example shares in individual companies or barrels of oil. But some options contracts are based on underlying assets which are not traded directly – important cases include contracts on short term interest rates and equity indices. Here settlement has to be in cash – for example based on the difference between the FTSE-100 index when the option is exercised and the 'strike price' agreed earlier.

Futures contracts are generally agreements to buy or sell assets at a date in the future at a price decided now. As with options they can be extended to notional assets such as equity indices that are not themselves traded. But unlike with options contracts, holders of futures have to buy or sell at the expiry date of the contract. For a risk neutral investor the futures price reflects the present value of the weighted average of possible outcomes for the price of the underlying asset. Because investors in futures contracts will often wish to use options markets to hedge their risks, it is often convenient to construct options contracts as options on futures contracts. But because options contracts generally expire at the same time as the underlying futures contracts, they are, in turn, closely tied to the equity indices, interest rates etc that underlie the futures contracts.

Short sterling futures and futures options

A short sterling futures contract is a sterling interest rate futures contract that settles on the three-month sterling interbank (BBA LIBOR) interest rate prevailing on the contract's delivery date.³ A short sterling futures option is a European option on a short sterling futures contract.⁴ Short sterling futures options expire on the same dates as the underlying futures contract. But as the futures contract itself settles on the prevailing 3-month LIBOR rate on that expiry date, the three-month LIBOR spot rate is in effect the actual underlying asset of the option contract on the expiry date. Short sterling option and futures contracts are standardised and traded between members of the London International Financial Futures and Options Exchange (LIFFE).

FTSE 100 index options

A FTSE 100 index option contract, if exercised at expiry, settles on the value of the FTSE 100 index prevailing at the expiry date of the option. The contracts are European and are traded on the London International Financial Futures and Options Exchange (LIFFE).

For more information on these contracts please visit www.liffe.com

2 The nature of option implied pdfs: some issues

(i) Risk neutrality

Option and futures contracts are 'derivative' assets. That is, their value is dependent on the value of an underlying asset or security. Put differently, both a derivative and its underlying asset are subject to the same source of price movements or, equivalently, the same sources of risk. This means that if the underlying asset is traded, it may be possible to use that asset to replicate a derivative asset – i.e. to produce the same payoffs as the derivative asset. It can be shown that this can be achieved by combining, in the right proportions, the underlying asset with a risk free asset – a government bond for example.⁵ So there is a no-arbitrage relationship between the price of the derivative asset and the price (value) of the constructed portfolio – that is, the two must be equal for there to be no opportunity for market participants to earn a riskless profit.⁶ In this sense, the derivative has a unique price and this price should not depend on the risk preferences of investors. So the price of the derivative should be the same whether the derivative is valued by assuming market participants are risk neutral or risk averse.

³ In this case, because LIBOR itself cannot be bought and sold, a notional asset is constructed as 100 minus LIBOR on the delivery date. For more information about LIBOR see the British Bankers Association's (BBA) website: <http://www.bba.org.uk/public/libor/>.

⁴ A short sterling call (put) option allows the holder to buy (sell) a short sterling futures contract.

⁵ In practice this is difficult, as it requires continuous re-balancing of the weights attached to the underlying and the risk-free assets in the portfolio. For a full discussion see Hull (2000), chapter 9.

⁶ In theory, if the two were unequal then by selling the higher priced asset/portfolio, buying the lower priced and holding the two positions until expiry, an investor could make a risk free profit.

This line of reasoning greatly simplifies the task of pricing derivative assets. Derivative assets like financial assets are priced by evaluating the expected *future* payoff from holding the asset. The expected future payoff must then be discounted to express it in current prices. Valuing a derivative in this way requires a view on the expected rate of return of the asset. In a risk neutral world, the expected rate of return for all assets is the risk free interest rate, which is known (or at least can be approximated very closely).⁷

Although the price of derivatives should be the same in a risk-neutral as in a risk-averse world, the interpretation to be put on information inferred from derivative prices may well differ in the two cases. Bank of England implied pdfs are backed out from option prices by using the risk free rate of interest as the expected rate of return. As such the information that we obtain from the pdf reflects market expectations in a risk neutral world. However it is generally accepted that investors are risk averse and will potentially have different expectations to those in a risk neutral world. The most obvious effect on implied pdfs of a change from a risk-neutral world to a risk-averse world is on the mean of an implied pdf. The mean of an implied pdf is equal to the futures price of the underlying asset. Futures prices are risk-neutral prices and have been shown to be biased expectations of actual future spot prices. The difference is attributed to a risk premium. This risk premium is necessary to compensate risk-averse investors for the riskiness of the underlying asset. So one of the implications of extracting implied pdfs from option prices by assuming investors are risk neutral is a lower mean than would be the case in a risk averse world. Turning to the higher moments of implied pdfs (dispersion, asymmetry and kurtosis), work thus far suggests that, outside of periods of extreme market turbulence, the assumption of risk neutrality has little impact.⁸ Overall, this means that the risk neutral assumption seems to be more important for the location of an implied pdf than it is for the shape of the distribution. As a result the risk neutral probabilities that we extract for different levels of the underlying asset in the future will be higher than those actually held by market participants.

(ii) *Fixed expiry vs. constant maturity*

Options contracts traded on exchanges have fixed expiry dates. Each day, the implied pdfs that are backed out from these contracts tell us something about market views of the possible change in the underlying asset price between that day and the expiry date. This means that from day to day the horizon over which we look ahead gets closer. This matters when we compare movements over time in the shape of the implied pdfs. One example is the dispersion or standard deviation of the implied pdf. This is interpreted as the expected volatility of the asset price over the remaining time to expiry. In the absence of any unexpected shocks, one would expect volatility to decline the closer we get to the expiry date.⁹ Even if the market's uncertainty is unchanged, measures derived from fixed-expiry

⁷ This is in contrast to a risk averse world where the expected rate of return on an asset is subjective and differs across investors.

⁸ See Bliss and Panigirtzoglou (2003) for example.

⁹ For serially uncorrelated data, volatility should decline at a rate given by the square root of the time to maturity. Thinking in terms of variances (variance is the square of volatility and is linear or additive in time) the variance over one month is

contracts would be expected to show a decline. To get around this, the Bank of England also estimates implied pdfs for a hypothetical option contract with a constant maturity.¹⁰ So, for example, a three-month constant maturity implied pdf always looks three months ahead each day. Movements in summary statistics from these implied pdfs are now free of the time-to-maturity effect inherent in those of the fixed expiry implied pdfs. For this reason, the implied pdf spreadsheets on this site show only very recent data for the first and second traded contracts/two traded contracts closest to expiry but show a much longer run of historical information for constant maturity implied pdfs. More information on the estimation of constant maturity implied pdfs can be found in the Bank of England *Quarterly Bulletin* article of Clews, Panigirtzoglou and Proudman.

3 Information on market expectations from option implied pdfs

Implied probability density functions can provide us with some information about a number of aspects of market expectations about future asset prices/interest rates. This information is primarily reflected in the shape of the distribution. Some illustrations of probability density functions and how changes in their shapes may be interpreted and related to market expectations are provided below. We focus on implied pdfs for short sterling interest rates (3-month LIBOR). Before turning to these we highlight some general features of a pdf and relate it to another commonly used method of expressing probability - the cumulative distribution function (cdf).

3.1 Probability density functions and cumulative distribution functions

Asset prices or interest rate levels could, in theory at least, take any value from zero to infinity. In practice, this range is too big and the outcomes that are viewed as likely will form a small subset of it. Nevertheless, the broad range that prices could take illustrates an important point – that is, that when considering the probability of an asset price being a specific value in the future, that specific price is just one of a possibly infinite number of values that it could be.¹¹ This is highlighted when looking at how probabilities attached to different asset price levels vary (or are ‘distributed’) over alternative price levels – the probability distribution function (pdf). Diagram 1 shows a pdf for short sterling interest rates on August 18, 2003. The x-axis shows future levels of short sterling interest rates and the y-axis the probabilities of these levels occurring. It is clear from the magnitudes of the probabilities on the y-axis that the probability of any individual level occurring in the next 6 months is very low. Because of the small probability attached to one particular level occurring, it is often more useful to look at probabilities attached to the asset price lying in a particular range. One idea is to look at the probability that the asset price will *at most* be a particular level – that is, the probability that the asset price level will be *less than or equal to* a specified price. This can be calculated from the pdf by

the sum of each of the daily variances. So, all else equal, if there are 30 days to expiry today, the volatility over the next thirty days should be higher than the volatility expected over the 23 days to maturity in 7 days time.

¹⁰ This hypothetical contract is constructed by interpolating across the prices of traded contracts with different times to maturity but with similar exercise prices.

summing up all of the area under the pdf curve up to the specified price level. Diagram 1 illustrates this, where the probability that short sterling interest rates will be *at most* 4.2% in 6 months time is given by area A. The total area under the pdf curve must sum to one. Continuing this idea, it follows, for example, that the probability that an asset price will lie in a specific range is given by the area under the pdf curve, between the upper and lower bounds of that range. Looking at probabilities in this way, another well used probability distribution is that of the cumulative distribution function (cdf). The information in this distribution is complementary to that in the pdf. Diagram 2 shows the cdf corresponding to the pdf shown in Diagram 1. The y-axis now shows probabilities that the asset price will be *less than or equal to* those levels specified on the x-axis. Continuing the 4.2% short sterling level example above, the y-axis value in the cdf corresponds to area A under the pdf in Diagram 1. The spreadsheets on the Bank of England internet site show eleven percentiles from the pdf each day.¹² Plotting these percentiles together will provide a representation of the cdf.

Diagram 1: Option implied pdf for short sterling rates in 6 months as at 18/08/2003

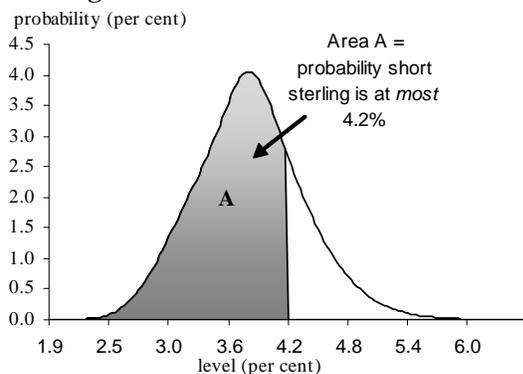
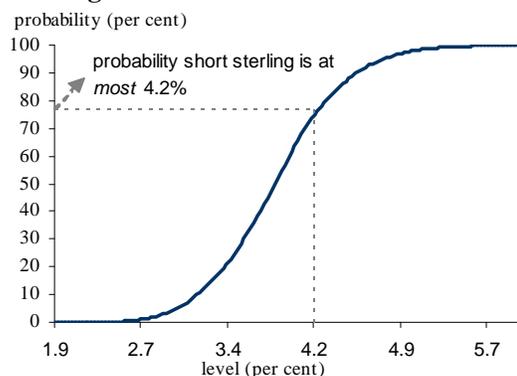
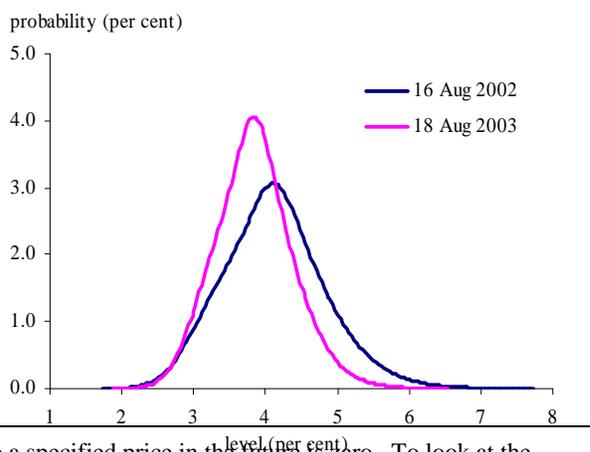


Diagram 2: Option implied cdf for short sterling rates in 6 months as at 18/08/2003



3.2 Market uncertainty

Diagram 3: Option implied pdfs for short sterling rates in 6 months



¹¹ In fact, the probability of an asset price being exactly equal to a specified price in the future is zero. To look at the probability of an asset price being a specific level we need to look at the probability that it will lie within a very narrow range around that specific level.

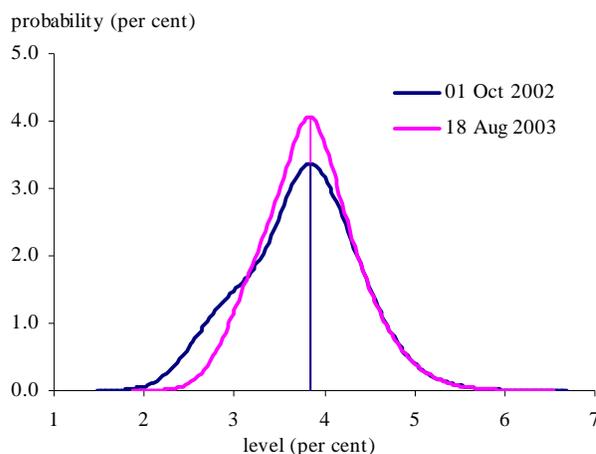
¹² A definition of pdf percentiles is provided in Section 4.

Changes in the width – or dispersion - of the distribution can inform us about changes in market uncertainty about future asset price levels. Diagram 3 compares the six month implied pdf for short sterling on 18 August 2003 with that of about one year earlier. The dispersion of the pdf decreased between the two dates so that the market attached non-zero probabilities to a narrower range of values in August 2003 than that of a year earlier. This suggests that markets were relatively less uncertain in August 2003 about future short sterling rates over the subsequent six months, than in August 2002. The standard deviation of the implied pdf and/or the option implied volatility are commonly used statistics to measure this dispersion or market uncertainty.

3.3 *Expected asymmetry*

The degree of asymmetry of the distribution can tell us about the market's assessment of the relative risks of future asset price moves in one direction relative to the other. Diagram 4 illustrates both symmetric and asymmetric pdfs. The August 2003 pdf suggests that the market attached very similar probabilities to outcomes above and below the mode.¹³ By contrast, the pdf of October 2002 puts more emphasis on interest rate levels below the mode than on those above it. This negative asymmetry suggests that the market assessment of the 'balance of risks' for future interest rates pointed toward expectations of lower, rather than higher, interest rates over the subsequent six months. The skewness of the implied pdf is a commonly used statistic to measure the degree of asymmetry.

Diagram 4: Option implied pdfs for short sterling rates in 6 months

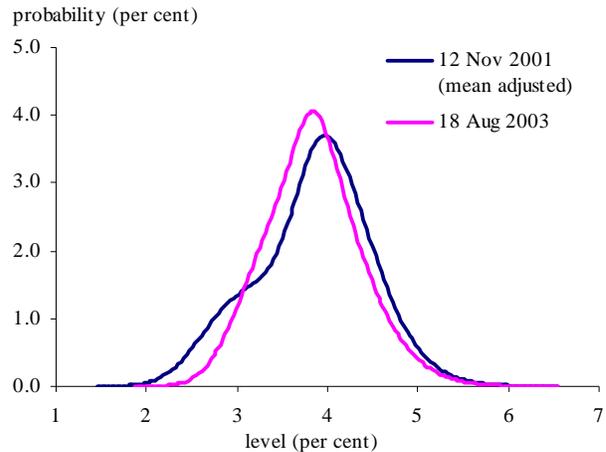


¹³ The mode is the interest rate level with highest probability of occurring and is defined more generally in Section 4.

3.4 Extreme movements

Finally, the amounts of probability attached to outcomes that are far away from current asset price levels – or the degree of ‘fatness’ of the tails of the pdf – can help us assess market expectations of the potential for extreme changes in asset price levels in the future. Diagram 5 compares the short sterling 6-month implied pdf at 18 August 2003 with an implied pdf from November 2001.

Diagram 5: Option implied pdfs for short sterling rates in 6 months



The (mean-adjusted) November 2001 pdf has much more probability density in the tails of the pdf – i.e. those regions far away from interest rates on those days (reflected in the centre of the pdf).¹⁴ These heavier tails are consistent with a market perception of a greater chance of large interest rate moves in the six months following November 2001, than in the six months after August 2003. Fatness of tails is usually measured statistically using the kurtosis of the implied pdf and/or a measure of the amount of probability in the tails of the implied pdf.

4 Summary statistic definitions

The summary statistics that are shown in the option implied pdf spreadsheets on the Bank of England internet site are defined as follows:

- i) *Mean*: the first moment of the implied pdf. It is a measure of central tendency or ‘centre of gravity’ for the implied pdf. Given the risk neutral nature of the implied pdfs, it is equal to the futures price of the underlying asset.
- ii) *Standard deviation*: the square root of the second moment of the implied pdf. It provides a measure of the dispersion of the implied pdf. It is not annualised and is expressed in the same units as the price of the underlying asset.
- iii) *Median*: the point of the implied distribution that has 50% probability above and below it. It is the 50th percentile and shows the level of the underlying asset that has a cumulative probability of occurring of 50%.
- iv) *Skew*: the third central moment of the implied pdf standardised by the third power of the standard deviation. It provides a measure of asymmetry for the distribution. It measures the relative

¹⁴ The implied on November 12, 2001 was shifted to bring its mean into line with that of the August 2003 pdf. This was for expositional purposes only.

probabilities (weighted by cubic distances) above and below the mean outcome, that is, the futures price. That the (cubic) distance from the central outcome (i.e. mean outcome) weights these probabilities is of particular importance. The difference between the unweighted probabilities above and below the mean has the opposite sign to that of skewness. For example, a pdf with positive asymmetry has a mean that is above the median and the mode. But the median divides the density into two parts of equal 50% probability mass. So, in this case, the unweighted probability above the mean is smaller than that below the mean.

- v) *Kurtosis*: the fourth moment of the pdf divided by the fourth power of the standard deviation. It provides a measure of how peaked the distribution is or, equivalently, the concentration of probability in the upper and lower tails of the implied pdf. A frequently used benchmark for kurtosis is that of a normal distribution which has a kurtosis of 3. It is location invariant and unitless.
- vi) *Xth Percentile*: the point of the distribution for which there is an x % probability for future values of the underlying being at/below this point (i.e. the cumulative probability of this asset price occurring). The spreadsheets show 11 percentiles – from the 5th to the 95th in steps of 10 plus the median. Putting all of the percentiles of an implied pdf together on any one day provides an estimate of the cumulative probability distribution function.¹⁵ Statistically the probability density function is given by the slope of the cumulative distribution function at each level of the asset price. That is, it is the change in the cumulative probabilities divided by the change in the corresponding asset price levels.

The summary statistics of the probability distribution of the level of the underlying asset may not always provide a useful view on market expectations. This can be due to substantial changes in the level of the underlying asset (e.g. FTSE 100 during the 1990's) or to the type of benchmark distribution for the level of an asset price commonly used in option pricing models. For example, if the level of an asset price is changing significantly it may be useful to consider the pdf of the proportionate change in the asset price, sometimes called the 'return' on the asset. In our framework, proportionate changes are measured by taking the difference between the logarithm of the current underlying futures price and the logarithm of each of a range of potential prices in the future. We then calculate the probabilities associated with each of these logarithmic changes to obtain an implied pdf for logarithmic changes in the level of an asset price. The standard deviation of this logarithmic returns pdf will be expressed in terms of proportionate changes in the underlying asset as opposed to the standard deviation from the level pdf which is measured in the same units as the asset price.

In considering asymmetry of market expectations, the skew of the logarithmic returns pdf may be preferable to that of the level pdf. The price of an asset cannot fall below zero, but is in principle unbounded on the upside. So many common option pricing models use naturally asymmetric

¹⁵ By contrast the probability distribution function (pdf) shows the implied probabilities of individual levels of the underlying asset price occurring. Strictly speaking the implied pdf shows the probability of lying within an arbitrarily small distance of each individual level.

distributions for the level of asset prices. And a useful point of reference may be an (asymmetric) lognormal distribution for asset prices, which however would imply that the logarithm of the asset price was normally distributed (with zero asymmetry). Although the pdfs we estimate are not the same as the ‘benchmark pdfs’ of the option pricing models, comparisons may most easily be made using the skew of our logarithmic return pdfs.

5 Data coverage

Our ability to estimate implied pdfs each day depends on a number of factors. These include the liquidity of the options markets for a particular asset and the range of exercise prices for which option contracts are traded. In estimating implied pdfs, the Bank of England uses quoted bid and ask prices as well as traded option prices. A number of conditions are set to ensure that a sufficient range of information and sufficient liquidity are available before an implied pdf is estimated. These include:

- a minimum number of exercise prices for each contract;
- a sufficiently wide range of exercise prices for each contract;
- in-the-money and deep-out-of-the-money options are not used for reasons of illiquidity¹⁶;
- contracts with less than five days to maturity are not used;
- contract prices must be convex and monotonic functions of corresponding exercise prices (i.e. satisfy basic theoretical conditions for option prices). Further they must produce probabilities which are non-negative and which sum to one (i.e. the area under the pdf curve equals one).

Missing values in the time series presented in the implied pdf spreadsheets are likely to be due to violations of one or more of the above conditions. Other factors that can result in days with missing values are bank holidays or lack of data from the exchanges concerned.

Short sterling futures options and FTSE 100 option contracts on LIFFE are both traded on a quarterly cycle – that is, with expiry dates in March, June, September and December of each year. Option contracts for FTSE 100 with expiry dates outside of the quarterly cycle are also available for trading. These FTSE 100 contracts, often referred to as ‘serial contracts’, have tended to exhibit more noise than their quarterly counterparts and are not used in extracting Bank of England implied pdfs. The spreadsheets for FTSE 100 and short sterling on the Bank’s internet site refer to the option contract

¹⁶ An option is referred to as ‘in-the-money’ (‘out-of-the-money’) if, given the strike price of the contract, it would (not) provide a positive gross payoff if exercised at the current underlying asset price. A call option, for example, is in-the-money (out-of-the-money) if the current underlying price is greater (less) than the strike price of the option. An out-of-the-money contract with a strike price that is far away from the current underlying asset price is referred to as a ‘deep’ out-of-the-money option.

with the closest expiry date in the quarterly cycle as the ‘first quarterly contract’ and to those with the next expiry date in the cycle as the ‘second quarterly contract’.

The range of data for the implied pdfs in the spreadsheet cover:

Short sterling	Constant maturity	3, 6 & 12 months ahead, 1988 – the present for 3 & 6 month, 1998 – the present for 12 month
	Fixed expiry	1 st and 2 nd quarterly contracts, data from most recent quarterly expiry date
FTSE 100	Constant maturity	3 & 6 months ahead, 1992 – the present
	Fixed expiry	1 st and 2 nd quarterly contracts, data from most recent quarterly expiry date

6 Acknowledgement & disclaimer

We are grateful to Bloomberg and to London International Financial Futures and Options Exchange for providing access to the underlying data used to calculate the option implied pdfs.

Every effort has been made to ensure this information is correct, but we cannot in any way guarantee its accuracy and you use it at your own risk.

Comments and questions can be directed to ImpliedPDFs@bankofengland.co.uk

7 References

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