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The size-centrality relationship in production networks

Nikola Dacic[1] and Marko Melolinna[2]

Abstract

We study two key characteristics of producers in a production network – size and centrality – and their relationship, which are intimately related to the extent of shock transmission in production networks, both at a macro and micro level. Our contributions are fourfold. First, we show empirically that the UK's production network has significant asymmetries in producer centrality, varies over time, and yields an empirical size-centrality relationship that tends to be positive in and outside of steady state. Second, we set up a static multisector model with a production network which allows us to link producer size and centrality to underlying shocks in the economy. We show that as long as input substitutability is less than unitary, technology shocks tend to induce negative (positive) co-movement between real output (Domar weights) and outdegrees, unlike preference shocks which tend to induce a positive size-centrality relationship. Third, we calibrate a dynamic model featuring a production network to UK data and use it to filter out technology and demand shocks. The implied size-centrality relationship from the filtered shocks confirms the intuition from the static model. Finally, we use this model to analyse the UK's post-2010 productivity growth slowdown from a production network perspective, distinguishing industries' accounting contributions from the contributions of industry-specific and common shocks. We find that idiosyncratic shocks to the manufacturing sector have played a key role in driving the aggregate productivity slowdown.

Key words: Business cycle, aggregate productivity, productivity puzzle, input-output linkages, production network.

JEL classification: E23, E24, E32.

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1 Introduction

The production of goods and the provision of services in modern economies typically involves inputs that firms source from other producers, in addition to the value-added they generate themselves. The existence of such input linkages gives rise to production networks, which exist at various levels of aggregation such as plant-level, firm-level, industry-level, and country-level. These production networks are very important in understanding how shocks transmit through the economy. A shock to a producer that has a very peripheral position in the network might have very muted transmission, unlike a shock to a central supplier from which many other producers source inputs. For example, an adverse supply shock to an important supplier of cement could negatively impact construction companies, which could then in turn negatively affect the real estate industry, and so on.

In this paper, we focus on two key characteristics of producers in a production network: their size and centrality. We make use of conventional notions of each characteristic. We measure producer size either in real (and absolute) terms by the level of each producer's real output, or in nominal (and relative) terms by the producer's Domar weight given by the ratio of its sales to nominal GDP. Regarding centrality measures, which are relative by construction, we focus on the first-order weighted outdegree (or simply 'outdegree'), which for a given producer is equal to the sum of shares of other producers' input consumption that it supplies.

Having defined our notions of producer size and centrality, our primary focus in this paper is on understanding the relationship between them. First, do large producers also tend to be central? Whilst one might expect this to be the case, it is not inconceivable that there exist large producers which primarily supply final goods or services, and thus have low centrality. In the context of the models we consider in the paper, this average association between the levels of size and centrality is related to the steady-state size-centrality relationship. Second, given a shock to a producer, do its size and centrality tend to change and, if so, do they change in the same direction? Again, whilst one might expect that they do, we show that the answer fundamentally depends on the nature of the shock. More specifically, technology shocks tend to imply a negative relationship, while demand shocks tend to imply a positive one. In the context of the models we consider, this refers to the (dynamic) size-centrality relationship away from the steady state. To our knowledge, this is the first paper to specifically analyse this relationship in a production network.

Producer size and centrality (and their relationship) are important in understanding both the aggregate and microeconomic effects of shocks. Regarding the former, the seminal paper by Hulten (1978) demonstrated that the first-order aggregate impact of microeconomic shocks is directly proportional to producer *size* (as measured by the producer's

Domar weight) in efficient economies. While the production network is irrelevant up to first-order in such settings, Baqaee and Farhi (2019) show that producer *centrality* will generally affect the second-order aggregate impact of microeconomic shocks. Moreover, Baqaee and Farhi (2020) show that in absence of perfect competition and frictionless markets, Hulten's theorem does not apply even to first-order. Therefore, the aggregate impact of microeconomic shocks depends both on producer size and centrality beyond the first order in efficient economies, and even at the first order in inefficient ones. Focusing instead on the microeconomic effects of shocks to a given producer on other producers, one needs to know the production network and such effects generally depend on both producer size and centrality (see Carvalho and Tahbaz-Salehi, 2019). The size-centrality relationship is thus intimately related to shock transmission in production networks, both at a macro and micro level.

In addition to analysing the empirical and model-implied relationship between size and centrality, we also attempt to shed new light on the UK's productivity growth slowdown following the global financial crisis of 2008-09 by casting it into a production network context in which producer size and centrality play a role. Several existing papers have focused on decomposing the UK productivity growth 'puzzle' in an accounting sense. For example, Tenreyro (2018) offers evidence that attributes around 75% of the slowdown to two sectors: finance and manufacturing.[1] Whilst insightful, such analyses do not identify the underlying shocks nor do they distinguish idiosyncratic as opposed to common shocks as potential drivers of the growth puzzle. In other words, do the observed slowdowns in manufacturing and finance—which are both relatively large and central sectors—reflect manufacturing-specific and/or finance-specific shocks? Or do they reflect common shocks or, perhaps, shocks to other industries with which manufacturing and finance have significant linkages? We aim to shed light on these questions.

Our contributions to the literature are the following. First, we analyse the characteristics of the UK economy's input-output network, showing that there are significant asymmetries in the degree of importance of industries as input suppliers. To our knowledge, this is the first paper to do so using UK data. We show that there is significant time-variation in the input-output network over time, which is inconsistent with a Cobb-Douglas aggregation of intermediate inputs in production. Second, we set up a simple multisector model in which intermediate inputs are aggregated using a constant elasticity of substitution (CES) aggregator and which features both supply (technology) and demand (preference) shocks. We show that if there is a reasonable complementarity across intermediate inputs to production, technology shocks generate a negative relationship between size (as measured by real output) and centrality, which is inconsistent with the empirical evidence. We show, uniquely, that demand shocks can help to reconcile the model's

[1] See also Riley et al. (2018), who reach a similar conclusion.

predictions with the data. Third, we consider a more general model with a production network calibrated using UK data, which we use to filter out technology and demand shocks. The implied size-centrality relationship resulting from the filtered shocks confirms our findings from the simple model. We argue that as long as the elasticity of substitution across intermediate inputs is less than unitary, demand-type shocks are needed in order to reconcile the model implied size-centrality relationship with its empirical counterpart. Fourth, we use this model to analyse the UK's productivity growth puzzle through a production network perspective and establish novel findings about its underlying drivers, with a key role for industry-specific shocks emanating in the manufacturing sector in driving the bulk of UK's post-2010 slowdown in productivity growth.

Related Literature Acemoglu et al. (2012) reinvigorated the interest in the role of production networks in the transmission of common and idiosyncratic shocks, albeit there is a long literature preceding this study (e.g. Long and Plosser, 1983, and Dupor, 1999). The key insight in this paper is that, in the presence of (intersectoral) input-output linkages, microeconomic idiosyncratic shocks may lead to aggregate fluctuations. Their results suggest that sizable aggregate volatility is obtained from sectoral idiosyncratic shocks only if there exists significant asymmetry in the roles that sectors play as suppliers to others. In production networks, complementarities and substitutability among production inputs typically play a key role (e.g., Jones, 2011). Atalay (2017) shows that elasticities of substitution (e.g. among intermediate inputs, or between value-added and intermediate inputs) have a significant effect on the importance of idiosyncratic (as opposed to common) shocks in driving aggregate fluctuations. Crucially, the existence of production networks makes it more difficult to disentangle truly common components of shocks from network-transmitted idiosyncratic shocks: a high degree of comovement in producers' output growth may arise both if common shocks are relatively important (even if the elasticities of substitution across producers as input-suppliers are high) and if common shocks are relatively unimportant provided that producers cannot easily substitute away across their input suppliers, making the two possibilities observationally similar.

Outline The rest of the paper is organised as follows. Section 2 describes the data that we use and analyses various features of the UK input-output network, providing several stylized facts. Section 3 develops a simple model that we use to characterise the implications of standard production network models for the size-centrality relationship. Section 4 uses a more general model that allows us to analyse the implied size-centrality relationship in a setting that is calibrated to match the UK input-output network. Section 5 applies this model to analyse the post-crisis productivity growth slowdown in the UK. Section 6 concludes.

2 Data and Stylized Facts

2.1 Data and Definitions

To analyse the key features of the UK production network, we will focus on the supply and use tables that contain data on input-output linkages between industries.[2] In Appendix A, we describe the data used in this paper.

The input-output supply and use tables contain, for a given year, an $N \times N$ matrix of intermediate input flows between industries, where N is the number of industries. Typically, rows represent input suppliers and columns input consumers (with every industry appearing as each respectively). By dividing each entry in the table by the sum in the respective column, we obtain shares that sum to one in each column and that correspond to the fraction of industry i's inputs produced by industry j (where i is a consuming and j a producing industry). We will refer to this matrix containing intermediate input shares as the *weighted input-output matrix*, $\mathbf{W}_t$, with a typical element ω_{ijt}. For later use, we also define the *adjacency input-output matrix*, $\mathbf{A}_t$, as the matrix whose entries a_{ijt} are a binary variable equal to one if $\omega_{ijt} > \tau$ and zero otherwise, for some chosen threshold τ.[3]

2.2 Stylized Facts About the UK Production Network

We now turn to analysing the weighted input-output matrix, $\mathbf{W}_t$.

2.2.1 Weighted Indegrees

First, we can analyse the heterogeneity in industries' role as input purchasers by multiplying each entry in each column i by the share of total intermediate consumption to gross output in industry i. Then, each column will sum to the share of intermediate consumption in gross output in the respective industry, which Acemoglu et al. (2012) refer to as the industry's *weighted indegree*. Figure 1 shows the nonparametric estimate of the empirical density of weighted indegrees of UK industries. Although the distribution has changed over time, it is generally relatively symmetric and centered around 0.4–0.6, with the majority of industries having an indegree at most one standard deviation from the mean.[4] Of course, industries may differ in terms of the distribution of their overall input purchases across suppliers (e.g. whether they source most of their inputs from a few or many suppliers). Nonetheless, this suggests that industries are relatively similar in their overall role as

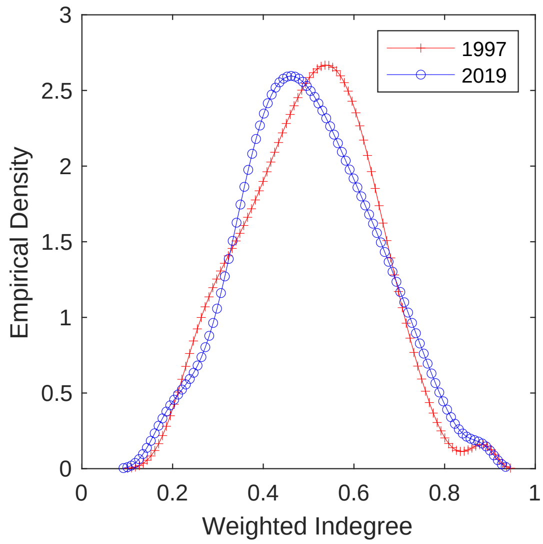
[2] No comprehensive firm-level dataset on input-output linkages is available for the UK.

[3] Below, we set $\tau = 1\%$, the same parametrisation as in Acemoglu et al. (2012).

[4] Acemoglu et al. (2012) obtain similar findings for the US: the mean weighted indegree in the US is around 0.55, with 71% of the industries' weighted indegrees being at most one standard deviation from this mean.

input purchasers in that most industries purchases inputs that amount to around 50–60% of their gross output.

Figure 1. Empirical densities of weighted indegrees in the UK



2.2.2 Weighted Outdegrees

The heterogeneity across industries in their role as input suppliers can be conveniently summarised by looking at the empirical *weighted outdegree* distribution. If this distribution is highly positively-skewed, this would suggest that there are many industries supplying highly-specialised inputs (to few other industries), and a few general-purpose input suppliers that provide inputs to many other industries. Recall that we denote by ω_{ijt} the share of industry i's inputs that are produced by industry j at time t. Then, the *first-order weighted outdegree* and *second-order weighted outdegree* of industry j are respectively defined as:

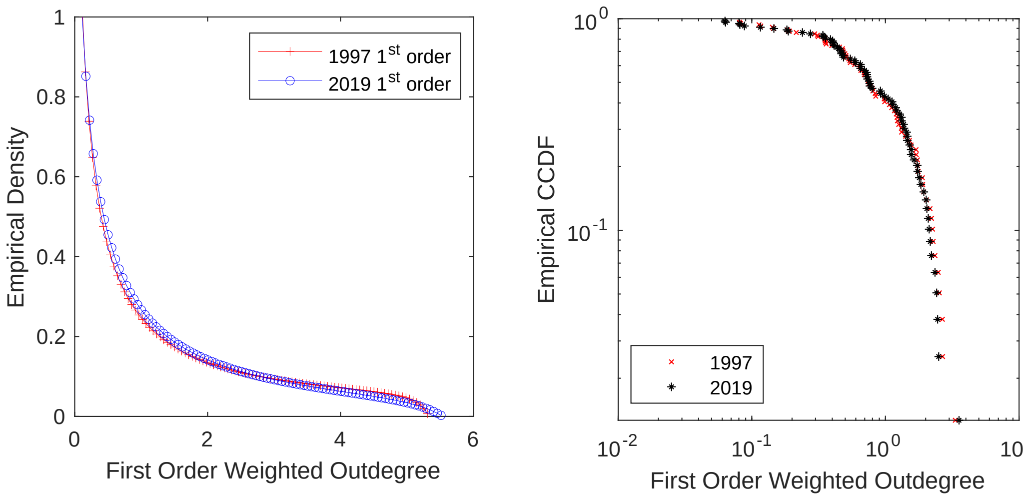
$$D_{jt}^{1,out} = \sum_{i=1}^{N} \omega_{ijt}, \quad \text{and} \tag{1}$$

$$D_{jt}^{2,out} = \sum_{i=1}^{N} \omega_{ijt} D_{it}^{1,out}. \tag{2}$$

Note that the first-order weighted outdegree measures the importance of a given industry as a *direct* input supplier.[5] The second-order weighted outdegree weights the input shares by the first-order weighted outdegree of the *consuming* industry and is thus the first step

[5]The first-order weighted outdegrees equal the sum over all the weights of the network in which industry j appears as an input-supplying industry. Therefore, this measure ranges from 0 if an industry does not supply inputs to any other industries, to N if a single industry is the sole input supplier of every industry in the economy.

Figure 2. Empirical density and counter-cumulative distribution of weighted outdegrees in the UK



towards measuring *indirect* linkages.[6]

Figure 2 (left panel) shows the empirical density function of (first-order) weighted outdegrees. In contrast to the empirical density functions of weighted indegrees, the distributions of outdegrees are skewed, with relatively heavy right tails (the empirical densities of second-order weighted outdegrees, not shown here, are similarly fat-tailed).[7] Relatedly, the right panel in Figure 2 shows the empirical counter-cumulative distribution function (CCDF) of weighted outdegrees.[8] The horizontal axis is the weighted outdegree for each industry (shown on a log scale), and the vertical axis (also shown on a log scale) gives the probability that an industry has an outdegree larger than or equal any correspond x-axis value. The linearity of the rightmost part of the distribution, in which industries with the largest weighted outdegrees lie, suggests that it is well-approximated by a power law distribution.[9] In other words, a small number of input suppliers are responsible for supplying the bulk of intermediate inputs.

[6]For instance, if two industries only supplied inputs to one other industry each, and if the respective shares of those industries' inputs attributable to these two industries were the same, then the two supplying industries would have equal first-order outdegrees. But if one of the industries supplied its inputs to an industry that is an important input supplier (and thus has a high outdegree) and the other supplied its inputs to an industry producing only final goods, the former's second-order outdegree would be larger than its first-order outdegree and the latter's second-order outdegree would instead be zero.

[7]The skewness of the empirical densities in Figure 2 is lower than that in Acemoglu et al. (2012), which is unsurprising given that they have around 400 industries in their dataset. Importantly, the skewness does not vanish as the number of industries increases, i.e. even at a high level of disaggregation some industries supply a large fraction of all inputs.

[8]The overall skewness in the distribution of weighted outdegrees may reflect both skewness along the extensive margin (number of supplying linkages a producer has) and the intensive margin (how important the producer is as a supplier to another). Empirically, the vast majority of the skew in the distribution of weighted outdegrees is driven by the intensive margin, i.e. relatively few suppliers are responsible for supplying the bulk of material inputs.

[9]Carvalho (2014) obtains similar findings for the US.

2.2.3 Bonacich Eigenvector Centrality

Measuring the importance of a node in a network using its weighted outdegree has its drawbacks. More specifically, this measure only captures *first-order* connectednness in that it only considers the immediate downstream customers. It may thus be considered to be the simplest centrality measure based on $\mathbf{W}_t$. Other centrality measures that are intended to capture higher-order connectedness (including that of a degree higher than two) exist and have been widely used. Typically, these measures embed the idea that a node's centrality (here, a node would correspond to an industry) is higher if its neighbours (here, other industries that it has input-output linkages with) are themselves well-connected.

Note that equations (1) and (2) can be written in matrix form as $D_t^{1,out} = \mathbf{W}_t\mathbf{1}$ and $D_t^{2,out} = \mathbf{W}_t^2\mathbf{1}$, respectively, where $\mathbf{1}$ denotes an $N \times 1$ vector of ones. Importantly, we could analogously define weighted outdegrees of higher orders. Bonacich (1987) introduced a centrality measure that is closely related to the weighted outdegrees of all orders. More specifically, given parameters β_1 and β_2, the *Bonacich eigenvector centrality* (B_{jt}) of industry j is defined as:

$$B_{jt}(\beta_1, \beta_2) = \sum_{i=1}^{N} (\beta_1 + \beta_2 B_{it})\, \omega_{ijt}. \tag{3}$$

The parameter β_1 is a normalisation parameter and only affects the length of the vector of Bonacich centralities, whereas the parameter β_2 reflects the degree to which an industry's centrality is a function of the centrality of those industries to whom it is connected. In matrix form, equation (3) is given by:

$$B_t = \beta_1 \mathbf{W}_t\mathbf{1} + \beta_2 \mathbf{W}_t B_t. \tag{4}$$

Solving for the vector of centralities B_t, we have that:

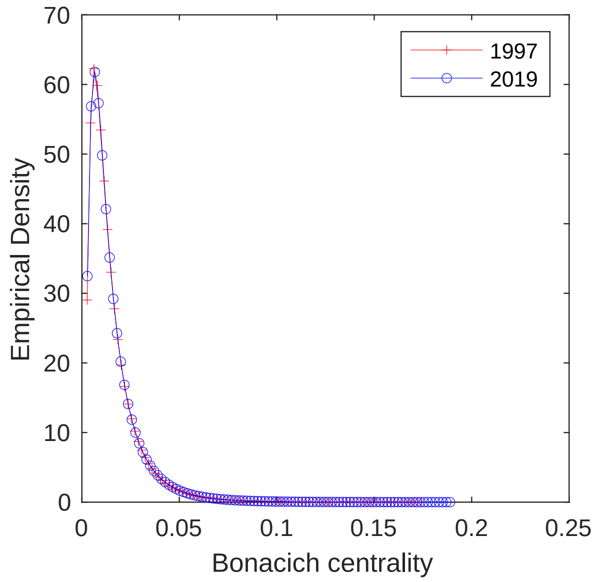
$$B_t = \beta_1 \left[\mathbf{I} - \beta_2 \mathbf{W}_t\right]^{-1} \mathbf{W}_t\mathbf{1}. \tag{5}$$

Using the power series expansion of the inverted matrix in equation (5), we have that the vector of Bonacich centralities is given by:

$$B_t = \left[\beta_1 \mathbf{W}_t + \beta_1\beta_2 \mathbf{W}_t^2 + \ldots\right] \mathbf{1}. \tag{6}$$

Therefore, the Bonacich eigenvector centrality is an infinite-order centrality measure in the sense that it captures both direct linkages (via first-order outdegrees) and indirect ones (via second-order outdegrees, third-order outdegrees, and so on). In other words, in our context, the Bonacich eigenvector centrality measure assigns to each industry a centrality score that is the sum of some baseline centrality level (common across all industries), and the centrality score of each of its downstream customers.[10] Figure 3 shows that the distribution of Bonacich eigenvector centralities is also fat-tailed, with the right tail of its CCDF (not shown) close to linear on a log-log scale, and thereby well-approximated by a power law. Note also that the right tail of the empirical density of Bonacich centralities is more prominent than that shown in Figure 2 for weighted outdegrees. Intuitively, once we take higher-order connectedness into account, that tends to further emphasise the role of central suppliers (in part because they also tend to supply inputs to more central industries).

Figure 3. Empirical density of Bonacich centralities in the UK



We can visualise the heterogeneity in industries' importance as input suppliers by graphically representing the UK input-output network. In Figure 4, each node corresponds to an industry and its size is proportional to the industry's Bonacich eigenvector centrality. The flows of inputs are represented by lines connecting the nodes, with the direction of the flow indicated by an arrow. Financial services (industry 64) are the most central input supplier in each year for which UK supply-and-use tables are available.[11] Several manufacturing industries are also very central, most notably manufacture of chemicals and chemical products (industry 20) and manufacture of basic metals (industry 24). Other very central industries include construction (industries 41-43), electricity, gas, steam and air conditioning supply (industry 35), computer programming, consultancy and related activities (industry 62), and employment activities (industry 78).[12] The majority of

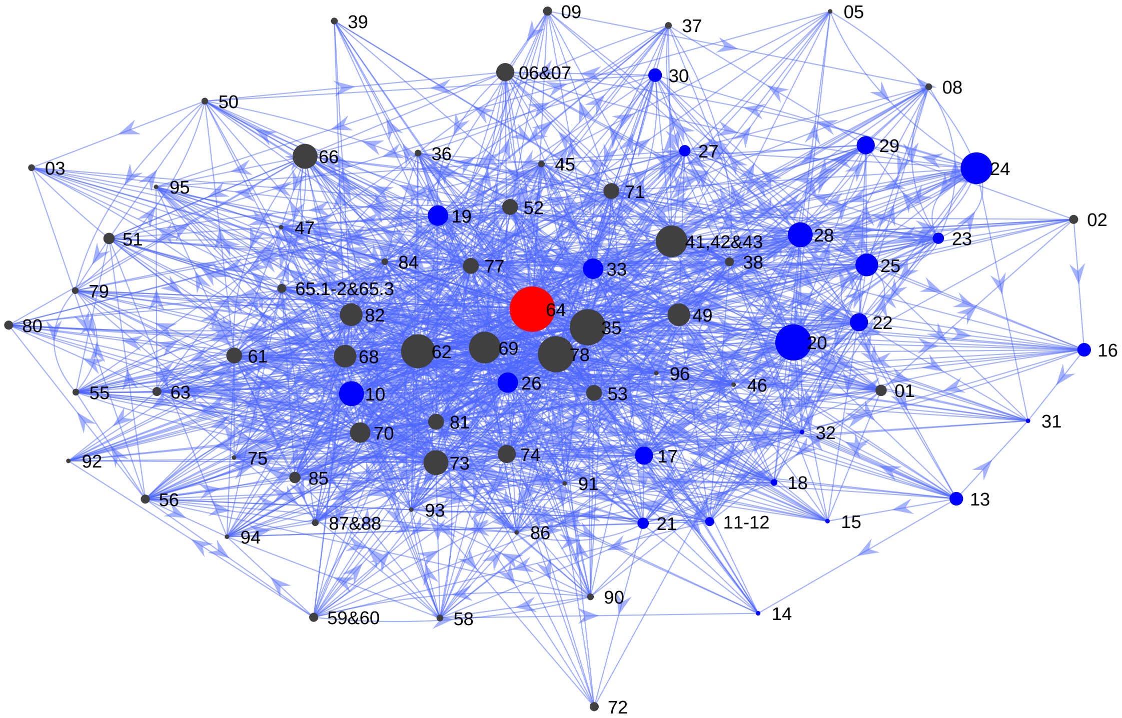
[10]Note that we set $\beta_1 = 0.5/N$ and $\beta_2 = 0.5$, where 0.5 is the share of intermediate inputs in production

[11]These financial service activities exclude insurance and pension funding. The UK's supply-and-use tables are available over 1997-2019.

[12]Employment activities (industry 78) include activities of employment placement agencies and other

industries are, however, much less important as input suppliers to other industries. This corroborates the earlier claim that most industries supply highly-specialised inputs (to few other industries) whilst a few general-purpose input suppliers provide inputs to many other industries.

Figure 4. A graphical representation of the UK input-output network in 2019



Notes: Finance (64) is the red node, manufacturing industries (10-33) are blue nodes.

2.2.4 Clustering in the Production Network

Beyond a visual inspection of Figure 4, we can also analyse more formally the extent to which industries cluster in the input-output network. The clustering coefficient is a standard network topology measure that captures the tendency to which nodes in a graph (in our case, industries in the input-output network) tend to cluster together. We define the average clustering coefficient as:

$$AverageClustering_t = \frac{1}{N}\sum_{i=1}^{N}\frac{\frac{1}{2}\left[\left(\mathbf{W}_t + \mathbf{W}_t^T\right)\left(\mathbf{A}_t + \mathbf{A}_t^T\right)^2\right]_{ii}}{s_{i,t}^{tot}\left(d_{i,t}^{tot} - 1\right) - 2s_{i,t}^{\leftrightarrow}}, \tag{7}$$

where $s_{i,t}^{tot} = (\mathbf{W}_t^T + \mathbf{W}_t)_i\mathbf{1}$, $d_{i,t}^{tot} = (\mathbf{A}_t^T + \mathbf{A}_t)_i\mathbf{1}$, and $s_{i,t}^{\leftrightarrow} = (\mathbf{W}_t\mathbf{A}_t + \mathbf{A}_t\mathbf{W}_t)_{ii}/2$, and where the subscript i (ii) denotes the ith row (iith entry) in the corresponding matrix, and $\mathbf{1}$

human resources provision.

denotes an N-dimensional vector of ones.[13] In a 'star economy', with one producer sourcing inputs from many other producers who are not themselves connected, the clustering coefficients would all be 0. On the other hand, if each industry sourced a fraction equal to $1/(N-1)$ of its inputs from *all* others (excluding itself), then all clustering coefficients would equal 1. Using our data, we find that the average clustering coefficient is very stable over time, with a mean of 0.84, suggesting that the UK's input-output network has a very high level of clustering.[14] More formally, this confirms the visual observation from Figure 4 that most 'triangles' (sets of three interconnected industries) include industries with higher centrality.

2.2.5 Stability of Input-Output Linkages

We also analyse the stability of network linkages. One might expect that the stronger a linkage (i.e. the larger is ω_{ijt}) and/or the more stable it is (i.e. the smaller the variance of ω_{ijt}), the greater the potential that a shock will be transmitted via that particular linkage. The (in)stability of a linkage can be related to the degree of substitutability among intermediate inputs. For instance, if all elasticities of substitution are unitary (i.e. the economy is Cobb-Douglas), all factor income shares are time-invariant so the variance of ω_{ijt} would be zero. If instead the elasticity of substitution between intermediate inputs is zero for all industries (i.e. if intermediate inputs are perfect complements) and the elasticity of substitution between value-added and intermediate inputs is zero, all intermediate inputs would be used in a constant proportion to output, thus implying that the variance of output would be proportional to the variance of input shares.

To assess the stability of $\mathbf{W}_t$, we compute the standard deviation of its entries, $\sigma(\omega_{ij})$. Figure 5 shows a scatterplot of the mean of ω_{ij} over time against its standard deviation. We see a clear positive association—entries in $\mathbf{W}_t$ which are larger on average also tend to be relatively more volatile. This is inconsistent with a Cobb-Douglas economy, which implies a horizontal line in Figure 5, with $\sigma(\omega_{ij}) = 0$.[15] For this reason, the models we consider in this paper do not feature a (fully) Cobb-Douglas economy and allow for a time-varying production network.[16]

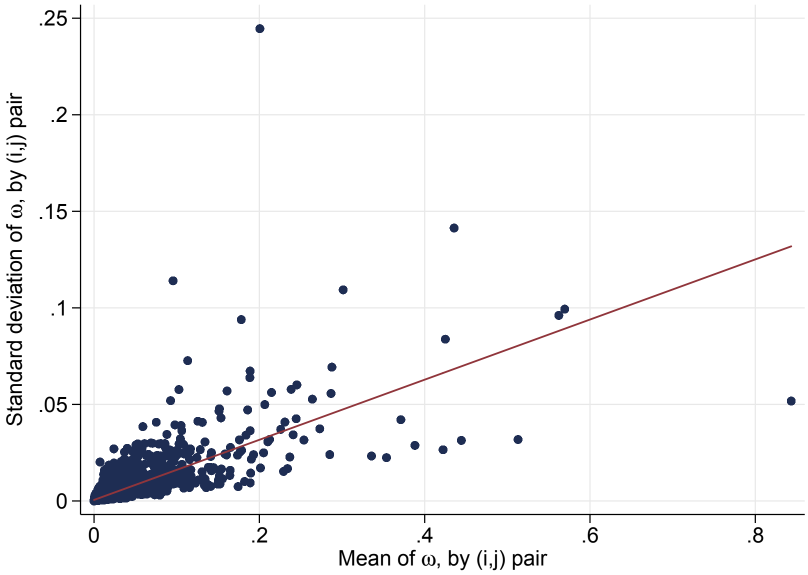
[13] This definition follows Clemente and Grassi (2017). Note that self-loops are not considered, i.e. the main diagonals in $\mathbf{W}$ and $\mathbf{A}$ are replaced with zeros.

[14] To put this into a perspective, Clemente and Grassi (2017) find that a network consisting of banks from the core 24 countries (as defined by the Bank for International Settlements (BIS)) has a clustering coefficient of around 0.9.

[15] Strictly speaking, complete stability in input-output shares is only consistent with a Cobb-Douglas aggregation of sectoral intermediates as long as sectors are not allowed to add/remove extra supplier sectors at the extensive margin. In particular, Acemoglu and Azar (2020) show how Cobb-Douglas aggregation can deliver time-varying input-output matrices through adjustment of supplier sectors at the extensive margin.

[16] To put the time variation of $\mathbf{W}_t$ into a perspective, the coefficient of variation (defined as the ratio of sample variance to sample mean) of the growth of outdegrees is larger than the coefficient of variation of real gross output growth for around 2/3 of all industries. By contrast, the coefficients of variation of $\mathbf{A}_t$

Figure 5. Scatterplot of mean and standard deviation of intermediate input shares



How does the time-variation in the input-output network relate to the correlation between different industries' output growth? Intuitively, one may expect that the higher the importance of industry j as a supplier of inputs to industry i (so the higher is ω_{ijt}) and/or the more stable this linkage is (so the lower is $\sigma(\omega_{ijt})$), the stronger the co-movement between the output of industries i and j. First, we estimate the following regression:

$$\rho(g_{ij}) = \alpha + \beta\overline{\omega}_{ij} + \varepsilon_{ij}, \quad i \neq j, \tag{8}$$

where $\rho(g_{ij})$ denotes the correlation between the growth rates of gross output in industries i and j and $\overline{\omega}_{ij}$ denotes the average share of inputs from industry j in industry i's total intermediate consumption. We find $\hat{\beta} = 1.38$, statistically significant at the 1% level. The pairwise correlations of gross output growth across industries tend to be higher the higher is the average share of inputs they source from each other: a 1pp increase in the average input share tends to be associated with close to a 0.014 increase in the correlation

based measures tend to be much smaller. Two such standard measures are industries' *degree*—defined as $\sum_{j=1}^{N} a_{ijt}$ for each industry i—and their *average path length*—defined as $\frac{1}{N-1}\sum_{i\neq j}^{N} ShortDist_{ijt}$ where $ShortDist_{ijt}$ denotes the shortest distance between industries i and j in year t.[17] The coefficient of variation of industries' average degree and average path length are lower than those of first-order weighted outdegrees for 65% and 90% of the industries, respectively.

coefficient of gross output growth rates.[18] Next, we consider the following regression:

$$\rho(g_{ij}) = \alpha + \beta\overline{\omega}_{ij} + \gamma\hat{\sigma}(\omega_{ij}) + \varepsilon_{ij}, \quad i \neq j, \tag{9}$$

where $\hat{\sigma}(\omega_{ij})$ denotes the sample variance of ω_{ijt}. We now find that $\hat{\beta}$ increases to 1.66 (and is still statistically significant at the 1% level), and $\hat{\gamma} = -1.82$ (and is significant at the 5% level). Intuitively, in equation (8), the coefficient on the average share of inputs ($\bar{\omega}_{ij}$) was likely biased downwards, which is due to higher shares also being more volatile (Figure 5) and higher volatility decreasing the correlation between output growth rates. In other words, once we have controlled for the average share of inputs industries source from each other, the pairwise correlations of gross output growth across industries tend to be lower for those pairs where the respective share of inputs is more volatile. Intuitively, insofar as the higher volatility of ω_{ijt} indicates greater substitutability across input suppliers, a high elasticity of substitution implies that even small changes in relative prices translate into large changes in input shares, increasing $\sigma(\omega_{ijt})$.

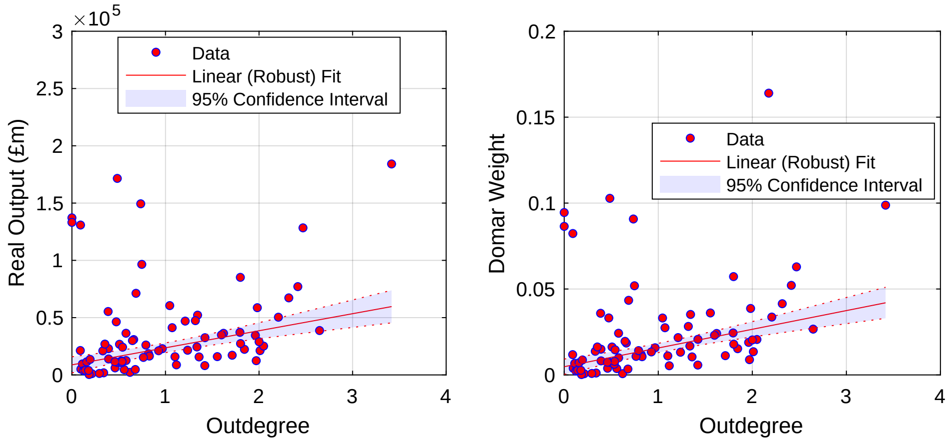
2.3 Empirical Size-Centrality Relationship

Our primary focus in this paper is on the relationship between the size and centrality of producers in a production network. Throughout, we use conventional notions of producer size and centrality, measuring real (and *absolute*) producer size using levels of real gross output, nominal (and *relative*) producer size using Domar weights, and centrality using (first-order) weighted outdegrees.

Figure 6 shows a scatterplot of the average levels of outdegrees against real output and Domar weights for all 79 industries in the UK over 1997-2019. There is a clear positive (and possibly non-linear) relationship, with larger industries also generally being more central. Importantly, since the sum of outdegrees across industries equals $N = 79$ in each year, there should be no concerns about a spurious relationship driven by, say, a common time trend.

[18] This finding is in line with Carvalho (2014), though based on $\mathbf{W}_t$ rather than the network distance measure Carvalho (2014) uses, which is based on $\mathbf{A}_t$.

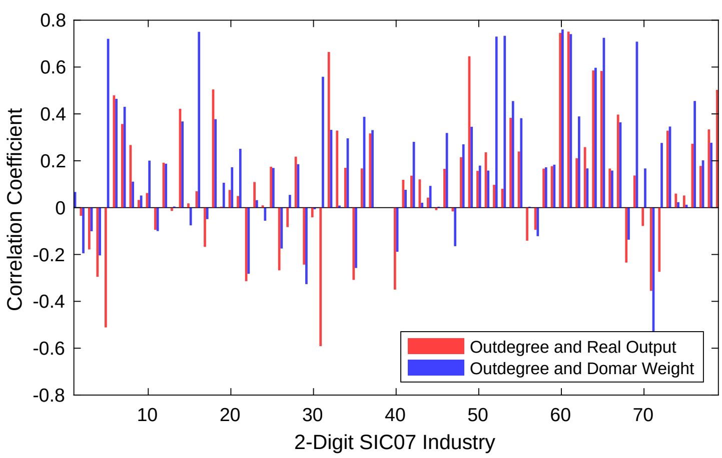
Figure 6. Scatterplot of the average levels of outdegrees and real output and Domar weights, by industry



Notes: All SIC07 2-digit industries included. Sample covers 1997-2019.

Focusing on each industry separately, Figure 7 shows the correlation between the growth rates of outdegrees and producer size (as measured by output or Domar weights). The correlation coefficient is positive for 68% (76%) of industries if real output (Domar weight) is the measure of producer size. In other words, on average and for many industries, as they grow in size, they also tend to become more central in the production network. For few industries, the opposite tends to be the case on average. As we argue in subsequent sections, this relationship is intimately related to the nature of shocks in the economy, particularly whether they are of a 'demand'-type (pushing quantities and prices in the same direction) or of a 'supply'-type (pushing the two in opposite directions).

Figure 7. Correlation between the growth rates of outdegrees and real output and Domar weights, by industry



Notes: Correlations are calculated between year-on-year growth rates. All 2-digit industries included. Sample covers 1997-2019.

From a modelling perspective, our interest is in (i) the model-implied size-centrality relationship in steady state (related to Figure 6), and (ii) the model-implied size-centrality relationship outside of steady state (related to Figure 7). Regarding the latter, in a Cobb-Douglas economy—in which all elasticities of substitution are unitary—the input-output network does not change in response to shocks, hence industries' centralities are time-invariant. In such an economy, Domar weights are also invariant to shocks. As soon as one deviates from the Cobb-Douglas benchmark—as we do in the subsequent sections—Domar weights and the input-output network may change in response to shocks.

Having provided evidence that the UK input-output network features highly asymmetrical producers in terms of their centrality and that the network exhibits significant time variation, we next set up a model which features a time-varying input-output network that we will use to analyse the implications of supply and demand shocks for the size-centrality relationship.

3 Static Model

Consider a static and closed multisector economy, with a representative household and perfectly competitive firms operating in N industries. The economy is static in the sense that agents do not make any dynamic choices, but this economy could still have a non-trivial temporal dimension depending on the time profile of exogenous shocks that hit it. The representative firm in each industry may source its intermediate inputs from any other industry, i.e. the economy features a production network. Importantly, each industry bundles its intermediate inputs using a CES aggregator. This is in contrast to the model in Acemoglu et al. (2012), which features a Cobb-Douglas economy, with unitary elasticities of substitution, which imposes a time-invariant production network, i.e. the weighted input-output matrix $\mathbf{W}_t = \mathbf{W}$ for all t. However, in the previous section, we showed that $\mathbf{W}_t$ exhibits significant time variation in the data. The simple model we set up can, for different sets of shocks in the model, generate a different $\mathbf{W}_t$.[19] Next, we set out the structure of the model economy.

3.1 Representative Household

The representative household has Cobb-Douglas preferences over N distinct goods (each produced by one of the N industries) summarised by the following utility function:

$$U(C_{1t}, ..., C_{Nt}, L_t) = \prod_{i=1}^{N} C_{it}^{\gamma_{it}} - \frac{L_t^{\phi}}{\phi}, \tag{10}$$

[19] Although the economy is static, we do not drop the time subscript so as to reflect that repeatedly observing this economy in the presence of varying shocks would generate time variation in the endogenous variables in the model.

where C_{it} is the consumption of good i, and $\gamma_{it} = \frac{D_{it}\xi_i}{\sum_{j=1}^{N} D_{it}\xi_i}$. The steady-state consumption share of good i is denoted by ξ_i and the exogenous shifters of households' preferences ('preference shocks') are denoted by D_{it}; clearly, $\sum_{i=1}^{N} \gamma_{it} = 1$. The representative consumer supplies labour L_t, with the disutility of labour parametrised by ϕ, and receives a wage W_t. The household's budget constraint is given by:

$$\sum_{i=1}^{N} P_{it} C_{it} = W_t L_t, \tag{11}$$

where P_{it} denotes the price of good i.

3.2 Producers

Let i be any one of the N perfectly competitive industries in the economy. Each industry produces a good that can be either consumed or used by other industries as an input for production. Gross output in each industry, Q_{it}, is produced using labour, L_{it}, and intermediate inputs, M_{it}, according to a Cobb-Douglas production function:

$$Q_{it} = \left(Z_{it} L_{it}^{\alpha_i} M_{it}^{1-\alpha_i}\right)^{\eta_i}, \tag{12}$$

where Z_{it} is an i.i.d. technology shock, α_i denotes the share of labour income in gross output, and η_i determines the returns to scale. If $0 < \eta_i < 1$, the production function exhibits decreasing returns to scale; if $\eta_i = 1$, the production function exhibits constant returns to scale. We assume that the intermediate input bundle, M_{it}, is a CES aggregate of the N intermediate inputs each industry sources from all of the other N industries (including itself):

$$M_{it} = \left[\sum_{j=1}^{N} \mu_{ij}^{\frac{1}{\varepsilon_M}} M_{ijt}^{\frac{\varepsilon_M - 1}{\varepsilon_M}}\right]^{\frac{\varepsilon_M}{\varepsilon_M - 1}}. \tag{13}$$

The parameters μ_{ij} correspond to the steady-state share of intermediate consumption from industry j in industry i's steady-state intermediate purchases from all other industries.

The key parameter in equation (13) is ε_M, which corresponds to the elasticity of substitution across the inputs in the intermediate bundle. Letting $\varepsilon_M \to 1$ turns the aggregator M_{it} into a Cobb-Douglas one; in that case, regardless of the values of shocks in the model (Z_{it} and d_{it}), the implied weighted input-output matrix $\mathbf{W}_t$ would be time-invariant. Since the gross output production function is a Cobb-Douglas aggregate of labour and intermediates, we know that for any sequence of shocks $\{Z_{it}, d_{it}\}$, the shares of labour and intermediate inputs in gross output remain unchanged. However, if $\varepsilon_M \neq 1$, the shares of individual intermediate inputs in gross output may vary with the shocks, resulting in a time-varying input-output network.

In Appendix B.1, we characterise the competitive equilibrium in this economy. Due to perfect competition, we must have $P_{it} = C'(Q_{it})$ in equilibrium, where $C'(Q_{it})$ denotes the marginal cost of producing Q_{it}. From producers' cost-minimisation problem, we thus obtain that in equilibrium:

$$P_{it} = \frac{Q_{it}^{\frac{1-\eta}{\eta}}}{Z_{it}} \left(\frac{W_t}{\eta\alpha_i}\right)^{\alpha_i} \left(\frac{P_{it}^M}{\eta(1-\alpha_i)}\right)^{1-\alpha_i}, \tag{14}$$

where P_{it}^M denotes the ideal price index associated with industry i's intermediate input bundle, M_{it}, and is given by:

$$P_{it}^M = \left[\sum_{j=1}^N \mu_{ij} P_{jt}^{1-\varepsilon_M}\right]^{\frac{1}{1-\varepsilon_M}}. \tag{15}$$

3.3 Equilibrium

In Appendix B.1, we show that the equilibrium of the model can be expressed as a system of $3N$ equations in $3N+1$ unknowns, $\{P_{it}, Q_{it}, L_{it}, W_t\}_{i=1}^N$. To solve the model, we take the consumer price index, P_t, as the numeraire and normalise it to equal 1 without loss of generality, which yields an additional equation. The CES aggregation of intermediate inputs does not allow for solving for the equilibrium analytically.[20]

Next, we derive our measures of producers' centrality and size in this economy.

3.4 Producer Centrality

We measure the centrality of each industry using its first-order weighted outdegree.[21] The first-order weighted outdegree of industry j is given by

$$D_{jt}^{out} \equiv \sum_{i=1}^N \omega_{ijt} = \sum_{i=1}^N \mu_{ij} \left(\frac{P_{jt}}{P_{it}^M}\right)^{1-\varepsilon_M}. \tag{16}$$

The first equality in equation (16) is definitional, and it implies that an industry becomes more central as long as the sum of (nominal) sales of intermediates it makes to other industries (including intra-industry intermediate sales) increases as a share of other

[20] Instead of considering a first-order approximation around a non-stochastic steady state, we also cross-checked our arguments made below by solving the model numerically, yielding results consistent with the arguments below.

[21] In Appendix B.3, we show that the conclusions we reach below also hold for other commonly used centrality measures (such as second-order weighted outdegrees). In other words, as long as $\varepsilon_M < 1$, industry j's second-order weighted outdegree (and, by extension, its Bonacich centrality) are also increasing in its price, P_{jt}.

industries' intermediate consumption. This therefore suggests that it is both quantities sold and prices of intermediates that determine industries' centrality. Importantly, the second equality—which follows from the equilibrium conditions in the model—instead suggests that the only equilibrium variables relevant for centrality are prices.[22]

In Appendix B.3, we show that the partial derivative of a given industry's outdegree with respect to its price is given by:

$$\frac{\partial D^{out}_{jt}}{\partial P_{jt}} = \sum_{i=1}^{N} (1-\varepsilon_M)\omega_{ijt}(1-\omega_{ijt})P_{jt}^{-1}. \tag{17}$$

Since $0 \leq \omega_{ijt} \leq 1$, the sign of this partial derivative is solely determined by ε_M. In the Cobb-Douglas benchmark ($\varepsilon_M = 1$), this derivative equals zero and ω_{ijt} equals μ_{ij} (see equation (16)), which is time-invariant. If $0 < \varepsilon_M < 1$ (i.e., there is a relatively low degree of substitutability in the intermediate bundle), ω_{ijt} is increasing in P_{jt}. By contrast, if $\varepsilon_M > 1$ (i.e., a relatively high degree of substitutability in the intermediate bundle), ω_{ijt} is decreasing in P_{jt}. Therefore, ε_M is crucial in determining the relationship between prices and producer centrality. In this economy, any time-variation in producer centrality will result from the variation in prices, in turn driven by technology and preference shocks.

To understand the intuition behind the relationship between prices and centrality, note that by substituting the optimal input demands for M_{ijt} and the bundle M_{it} in the definition of ω_{ijt}, we obtain the following relationship:

$$\omega_{ijt} = \frac{P_{jt}M_{ijt}}{P^M_{it}M_{it}} = \underbrace{\left[\frac{P_{jt}}{P^M_{it}}\right]}_{\text{`price' effect}} \underbrace{\left[\left(\frac{P^M_{it}}{P_{jt}}\right)^{\varepsilon_M} \mu_{ij}\right]}_{\text{`quantity' effect}}. \tag{18}$$

Suppose a shock decreases P_{jt}. All industries (including industry i) now want to substitute towards good j as it is cheaper, i.e. the quantity of inputs they source from industry j will increase. Since the price of any industry i's bundle of intermediate inputs, P^M_{it}, falls by less than P_{jt} (since good j is just one of the goods in the bundle, with a weight less than 1), the 'price' effect in equation (18) falls. If $\varepsilon_M = 1$, the 'quantity' effect will exactly offset the price effect and there will be no change in ω_{ijt}. If $\varepsilon_M < 1$, the increase in the quantity effect will less than offset the fall in the price effect—because industries do not easily substitute towards the now cheaper good j—and ω_{ijt} will fall, decreasing the centrality of industry j by equation (16). The opposite happens if $\varepsilon_M > 1$, in which case industries can substitute relatively easily across their intermediate inputs and will increase the quantity of good j they consume in a way that will more than offset the fall

[22]See Appendix B.2 for a derivation of equation (16). Note that the assumption of perfect competition guarantees that the representative firm in each industry charges the same price on the inputs that it sells to any other industry.

due to the price effect. Therefore, prices are a sufficient statistic to understand the effects of shocks on producer centrality in this model, and the parameter which fundamentally determines this relationship is ε_M.

Therefore, to understand how technology and preference shocks affect industries' centrality, we need to understand how prices respond to these shocks. In the model in this section, a closed-form solution for prices does not exist due to the CES aggregation of intermediate inputs. Instead of resorting to an approximate solution for the dynamics of this model around its steady state, we find its solution numerically in two example economies assuming $\varepsilon_M < 1$ (consistent with our empirical estimates presented in the next section) and using conventional values for the remaining model parameters. Our results confirm that prices—and thus producer centrality—positively respond to (own) negative productivity and/or positive preference shocks (see Appendix B.5). We also show that this result holds true under two generalisations around the elasticity of substitution across intermediate inputs, in particular to allowing for different values of ε_M across producers, and to allowing for heterogeneous elasticities of substitution across subgroups of intermediate inputs (see Appendices B.6 and B.7).

3.5 Producer Size

We consider two measures of producer size in the production network:

1. industries' real gross output (Q_{it}), and their
2. Domar weights (λ_{it}), defined as the ratio of industries' nominal gross output ($P_{it}Q_{it}$) to nominal GDP; nominal GDP is given by $\sum_{i=1}^{N}\left(P_{it}Q_{it} - P_{it}^{M}M_{it}\right)$.

The former measure is related to the real (and absolute) size of an industry in that an increase in industry i's real gross output need not entail a change in the other industries' size. This is in contrast to Domar weights, computed from nominal variables and relative by construction in that a change in industry i's Domar weight necessarily entails a change in at least one other industry's Domar weight.[23] We consider both real and nominal notions of producer size given the key role of prices in determining the size-centrality relationship.

How do our measures of size respond to technology and preference shocks? While any industry's real output and Domar weights may in principle respond to a shock to any industry, the majority of variation in producer size (and centrality) will tend to be driven by own shocks.[24] First, real output (Q_{jt}) always increases (decreases) in response

[23]Whilst Domar weights measure the importance of an industry as an *output* supplier to the entire economy, our chosen measure of centrality (first-order weighted outdegrees) measures the importance of an industry as an *input* supplier to the entire economy.

[24]We demonstrate this in Section 4 in a more general model that is calibrated to match the UK production network.

to positive (negative) shocks to technology (Z_{jt}). As long as an industry has a positive weight in the household's consumption basket (i.e. $\gamma_{jt} > 0$), then a positive (negative) preference shock will also increase real output of industry i. Second, to understand how Domar weights respond to shocks, note that we must have, by definition:[25]

$$\lambda_{jt} = \frac{P_{jt}Q_{jt}}{\mathrm{P}_t C_t} = \frac{P_{jt}C_{jt}}{\mathrm{P}_t C_t} + \frac{\sum_{i=1}^{N} P_{jt}M_{ijt}}{\mathrm{P}_t C_t}. \tag{19}$$

The first term after the second equality equals γ_{jt}, and the second term can be rewritten so as to yield:[26]

$$\lambda_{jt} = \gamma_{jt} + \sum_{i=1}^{N}(1-\alpha_i)\omega_{ijt}\lambda_{it}. \tag{20}$$

Therefore, in equilibrium, industries' Domar weights are a sum of their consumption shares and a weighted-average of all Domar weights in the economy, where the weights are a product of the shares of intermediates in output and intermediate input shares in the bundle of intermediates. Assuming that $\alpha_i = \alpha$ (for simplicity), we can write equation (20) in matrix form and solve for the Domar weights:

$$\lambda_t = \left[\mathbf{I} - (1-\alpha)\mathbf{W}_t\right]^{-1}\gamma_t. \tag{21}$$

Using the power series expansion of the matrix multiplying γ_t, we obtain:

$$\lambda_t = \left[\mathbf{I} + (1-\alpha)\mathbf{W}_t + (1-\alpha)^2\mathbf{W}_t^2 + ...\right]\gamma_t. \tag{22}$$

Note that by setting $\beta_1 = \beta_2 = (1-\alpha)$ in equation (6), we have that the Domar weights are equal to the sum of the vector of consumption shares (γ_t) and the vector of modified Bonacich centralities where in equation (22) the vector of ones is replaced by the vector of consumption shares. Intuitively, on top of the sales that an industry generates by selling final goods to the representative household, its total sales in equilibrium also include the sale of intermediate goods which ultimately get transformed into final goods sold to the household.[27]

Using equation (22), we can see that Domar weights will fall in response to positive technology shocks (and vice versa). A positive technology shock to industry j will leave γ_t unchanged (as it is pinned down solely by household preferences and preference shocks), but it will tend to decrease the entries in the jth row of the matrix $\mathbf{W}_t$, thereby lowering

[25]In this economy, nominal GDP—the denominator in the Domar weights—equals nominal aggregate consumption.

[26]See Appendix B.4 for a derivation of equation (20).

[27]In the extreme case where the production network contains no non-trivial linkages, so $\mathbf{W}_t = \mathbf{I}$, the Domar weights are given by: $\lambda_t = [1 + (1-\alpha)/\alpha]\,\gamma_t = \alpha^{-1}\gamma_t$.

Table 1: Effect of Shocks to an Arbitrary Producer j on its Centrality and Size

Variable	Positive Technology Shock ($Z_{jt} \uparrow$)	Positive Preference Shock ($D_{jt} \uparrow$)
Producer centrality (D_{jt})	$\downarrow$	$\uparrow$
Producer size: Domar weight (λ_{jt})	$\downarrow$	$\uparrow$
Producer size: real output (Q_{jt})	$\uparrow$	$\uparrow$

Notes: The effects shown correspond to a case where the elasticity of substitution across intermediate inputs is less than 1.
The signs of the effects get reversed in case of negative shocks.

the j-th entry in λ_t, i.e. industry j's Domar weight.

By contrast, equation (22) shows that a positive preference shock to industry j will increase the jth element of γ_t and reduce the other elements.[28] The effect of preference shocks on prices (and thus the matrix $\mathbf{W}_t$) in this (static) economy depends on the returns to scale (η_i). If there are constant returns to scale, $\mathbf{W}_t$ will not change in response to preference shocks. To the extent that there are decreasing returns to scale, a positive preference shock to industry j will tend to increase entries in the jth row of $\mathbf{W}_t$, further increasing industry j's Domar weight.

Table 1 summarises our conclusions. We find that technology shocks tend to contribute to a negative relationship between producer centrality and real output, and a positive relationship between producer centrality and Domar weights.[29] Demand shocks, on the other hand, tend to contribute to a positive size-centrality relationship.

Our analysis above suggests that, as long as the production network is time-varying and the elasticity of substitution among intermediate inputs is less than unitary, the canonical models of production networks featuring *only* technology shocks tend to imply a strongly negative relationship between producers' output quantity and centrality, inconsistent with our empirical evidence for the UK. In other words, an industry repeatedly hit by negative supply-side shocks would therefore become simultaneously progressively smaller yet more central, and vice versa. As we show, to our knowledge uniquely, introducing demand-type shocks is a way to reconcile the model-implied size-centrality relationship with its empirical counterpart. The model we introduce next is dynamic and features both supply and demand shocks.

[28]This is true by construction, since $\sum_{i=1}^{N} \gamma_{it} = 1$.

[29]Note that technology shocks may imply a positive size-centrality relationship for the industries *other than* the industry in which the shock originated. In practice, the majority of variation in industries' size and centrality will tend to be driven by their *own* shocks, hence our emphasis on the implied relationship for the 'shock-originating' industry.

4 Dynamic Model

We now consider a closed, multisector economy in which agents face dynamic optimisation problems, drawing heavily on the model from Atalay (2017). This model economy relaxes various potentially overly restrictive assumptions that were present in the simple model from the previous section: now, both the representative consumer and the representative firms in each industry make dynamic choices, industries also source capital inputs from each other, and various elasticities (other than that across intermediate inputs) are not necessarily unitary.

In this section, our contributions based on this model are threefold. First, we use it to filter out the technology and preference shocks from the quarterly data on value-added and price growth across all industries in the UK. By contrast, Atalay (2017) filters out technology shocks only.[30,31] Second, we derive the equations for our measures of producer size and centrality, and show the implied size-centrality relationship as a result of the filtered shocks. Third, through the lens of this model, we analyse the dynamics of UK productivity growth and disentangle the relative importance of sector-specific and common shocks.

We first provide a brief exposition of the model below.

4.1 Representative Household

The representative household derives utility from the N different consumption goods (produced by the N industries) and disutility from supplying labour. The lifetime utility is given by:

$$U_0 = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[\log \left[\sum_{i=1}^{N} \left(\frac{D_{it}\xi_i}{\sum_{i=1}^{N} D_{it}\xi_i} \right)^{\frac{1}{\varepsilon_D}} C_{it}^{\frac{\varepsilon_D - 1}{\varepsilon_D}} \right]^{\frac{\varepsilon_D}{\varepsilon_D - 1}} - \frac{\varepsilon_{LS}}{\varepsilon_{LS}+1} \left(\sum_{i=1}^{N} L_{it} \right)^{\frac{\varepsilon_{LS}+1}{\varepsilon_{LS}}} \right], \tag{23}$$

where β denotes the rate of time preference, ξ_i denotes the time-invariant differences in the importance of consumption goods in aggregate consumption, D_{it} denotes a preference shock to good i at time t, C_{it} denotes the final consumption purchases of good i at time t, and L_{it} denotes the supply of labour to industry i at time t. In steady state, $D_{it} = 1$ for all i and we assume $\sum_{i=1}^{N} \xi_i = 1$. The elasticities of substitution determine how easily the consumer substitutes across the different consumption goods (ε_D) and how responsive the

[30] Also, while we filter out such shocks here, we instead simulated them in Appendix B.5.

[31] Specifically, Atalay (2017) filters out demand shocks only for the government sector as part of the sensitivity analysis (in which case, he assumes away supply-side shocks in this sector). Instead, we retain supply-side shocks for the government sector while also allowing for demand-side shocks in all industries.

consumer's desired labour supply is to the prevailing wage (ε_{LS}).

4.2 Producers

Each industry produces a quantity (Q_{it}) of good J at time t using capital (K_{it}), labour (L_{it}), and intermediate inputs (M_{it}) according to the following production function:

$$Q_{it} = A_{it}^{\eta_i} \left[(1-\mu_i)^{\frac{1}{\varepsilon_Q}} \left(\left(\frac{K_{it}}{\alpha_i} \right)^{\alpha_i} \left(\frac{L_{it}}{1-\alpha_i} \right)^{1-\alpha_i} \right)^{\frac{\varepsilon_Q - 1}{\varepsilon_Q}} + \mu_i^{\frac{1}{\varepsilon_Q}} M_{it}^{\frac{\varepsilon_Q - 1}{\varepsilon_Q}} \right]^{\eta_i \frac{\varepsilon_Q}{\varepsilon_Q - 1}} . \tag{24}$$

As in the simple model, the parameter η_i parameterises the degree of returns to scale. The parameters μ_i and α_i reflect long-run averages in each industry's usage of intermediate inputs, labour, and capital. These parameters will eventually be inferred from the factor cost shares of each industry. A_{it} denote the factor-neutral technology level of industry i at time t. Note that A_{it} may be correlated in any arbitrary way across industries. The parameter ε_Q determines how easily value-added can be substituted with intermediate inputs.

The evolution of capital in each industry is given by:

$$K_{i,t+1} = (1 - \delta_K) K_{it} + X_{it}. \tag{25}$$

The capital stock is accumulated via an industry-specific bundle of investment goods, X_{it}, and depreciates at a rate δ_K, common across industries. The industry-specific investment bundle is produced by combining the goods produced by potentially all industries:

$$X_{it} = \left(\sum_{j=1}^{N} \left(\Gamma_{ij}^{X} \right)^{\frac{1}{\varepsilon_X}} \left(X_{ijt} \right)^{\frac{\varepsilon_X - 1}{\varepsilon_X}} \right)^{\frac{\varepsilon_X}{\varepsilon_X - 1}} . \tag{26}$$

The parameters Γ_{ij}^{X} determine how important industry j is as an investment-good supplier to industry i. The parameter ε_X parametrises the substitutability of various investment goods in the investment bundle.

The intermediate input bundle of industry J is defined analogously:

$$M_{it} = \left(\sum_{I=1}^{N} \left(\Gamma_{ij}^{M} \right)^{\frac{1}{\varepsilon_M}} \left(M_{ijt} \right)^{\frac{\varepsilon_M - 1}{\varepsilon_M}} \right)^{\frac{\varepsilon_M}{\varepsilon_M - 1}} . \tag{27}$$

Above, Γ_{ij}^{M} determines how important industry j is as an intermediate-good supplier to industry i, and ε_M parametrise how easily substitutable the various goods in the intermediate bundle are.

The market-clearing condition for each industry states that output can be used for consumption, as an intermediate input, or for investment:

$$Q_{jt} = C_{jt} + \sum_{i=1}^{N} (M_{ijt} + X_{ijt}). \tag{28}$$

4.3 Exogenous Processes

Let A_t denote the vector of factor-neutral technology shocks in the N industries, $(A_{t1}, ..., A_{tN})'$. Similarly, let D_t denote the vector of preference shocks in the N industries, $(D_{t1}, ..., D_{tN})'$. We assume the evolution of A_t and D_t follows a geometric random walk:

$$\log A_t = \log A_{t-1} + \omega_t^A, \tag{29}$$

$$\log D_t = \log D_{t-1} + \omega_t^D. \tag{30}$$

We do not impose any restrictions on any of the variance-covariance matrices.

4.4 Equilibrium and Log-Linear Equations for Size and Centrality

Since this economy satisfies the welfare theorems, it suffices to solve the social planner's problem.[32] Given the non-linear nature of the model, we consider a first-order log-linear approximation around the non-stochastic steady state (see Appendix C.1). The evolution of output can be written as:

$$\Delta \log Q_{t+1} = \mathbf{\Pi}_1 \Delta \log Q_t + \begin{bmatrix} \mathbf{\Pi}_2 \vdots \mathbf{\Pi}_4 \end{bmatrix} \begin{bmatrix} \omega_t^A \\ \omega_t^D \end{bmatrix} + \begin{bmatrix} \mathbf{\Pi}_3 \vdots \mathbf{\Pi}_5 \end{bmatrix} \begin{bmatrix} \omega_{t-1}^A \\ \omega_{t-1}^D \end{bmatrix}, \tag{31}$$

where the $N \times N$ matrices $\mathbf{\Pi}_1$, $\mathbf{\Pi}_2$, $\mathbf{\Pi}_3$, $\mathbf{\Pi}_4$, and $\mathbf{\Pi}_5$ are functions of the model parameters only.[33] The baseline approach in Atalay (2017) assumes away preference shocks (i.e. $\omega_t^D = 0$) and uses data on industries' output growth to filter out the technology shocks using equation (31). As in Foerster et al. (2011), the equilibrium in this model permits a VARMA(1,1) representation in industries' output growth rates if preference shocks are assumed away. Instead, we allow for preference shocks, so to filter all $2N$ shocks in the model, we instead use data on industries' value-added and prices, which admit a VARMA(1,1) representation as well (see Subsection 4.6 below).

[32]The competitive equilibrium of the model is derived in Appendix F of Atalay (2017).
[33]See Appendix C.2 where we log-linearise the model's equilibrium and solve for these matrices.

4.4.1 Producer Size

As in the simple model, we use industries' real output and Domar weight as measures of their absolute and relative size, respectively. Denote the vector of log-deviations of real output and Domar weights from their steady-state values as $\hat{q}_t$ and $\hat{\lambda}_t$, respectively. Denoting the vector of log-deviations of prices from their steady-state values as $\hat{p}_t$, we can write nominal gross output as $\hat{n}_t \equiv \hat{q}_t + \hat{p}_t$. Thus, Domar weights are defined as $\hat{\lambda}_t = \hat{p}_t + \hat{q}_t - \widehat{ngdp}_t \cdot \iota$ where $\widehat{ngdp}_t$ denotes the log-deviation of nominal GDP from its steady state, and ι denotes an $N \times 1$ vector of ones.

Appendices C.2 and C.5 show that our two measures of industries' size can be expressed as functions of the state variable (capital) and the shocks as follows:

$$\begin{bmatrix} \Delta \hat{q}_t \\ \Delta \hat{\lambda}_t \end{bmatrix} = \begin{bmatrix} \mathbf{\Phi}_k \\ \mathbf{\Lambda}_k \end{bmatrix} \Delta \hat{k}_t + \begin{bmatrix} \mathbf{\Phi}_a \\ \mathbf{\Lambda}_a \end{bmatrix} \omega_t^A + \begin{bmatrix} \mathbf{\Phi}_d \\ \mathbf{\Lambda}_d \end{bmatrix} \omega_t^D, \tag{32}$$

where the matrices in square brackets are functions of model parameters only.

4.4.2 Producer Centrality

As in the simple model, we consider first-order weighted outdegrees as our measure of producer centrality. Appendix C.4 shows that, as in the simple model, first-order outdegrees can be expressed as a function of prices only, namely $\hat{d}_t^o = \mathbf{D}\hat{p}_t$, where $\mathbf{D}$ is a matrix consisting of various terms involving the model parameters.[34] The log-deviation of first-order weighted outdegrees from their steady state values can be expressed as:

$$\Delta \hat{d}_t^o = \mathbf{D}_k \Delta \hat{k}_t + \mathbf{D}_a \omega_t^A + \mathbf{D}_d \omega_t^D. \tag{33}$$

In Appendix C.4, we show that as we approach the Cobb-Douglas benchmark ($\varepsilon_M \to 1$), the matrix $\mathbf{D}$ approaches the zero matrix. In other words, all intermediate shares become constant (i.e., $\mathbf{W}_t = \mathbf{W}$), and so do outdegrees.

4.5 Calibration

For the purposes of this paper, the key parameter is the elasticity of substitution across inputs in the intermediate bundle (ε_M). More specifically, as was the case in the simple model (see equation (16)), the value of ε_M will determine the relationship between prices and outdegrees, inter alia: as long as $\varepsilon_M < 1$, an idiosyncratic technology shock to an industry that increases its price will simultaneously increase its outdegree. In addition, as Atalay (2017) shows, this parameter will directly affect the relative importance of

[34]The difference is that in the simple model, this result was obtained as an analytical solution rather than as a first-order approximation.

industry-specific and common shocks as implied by the model filter: if ε_M is low (high), industry-specific (common) shocks will tend to be relatively more important in explaining aggregate GDP dynamics.

Before we proceed more formally in estimating ε_M, recall from equation (16) that $\omega_{ijt} = \mu_{ij} p_{ijt}^{1-\varepsilon_M}$, where $p_{ijt} \equiv P_{jt}/P_{it}^M$. As long as ω_{ijt}, μ_{ij}, and p_{ijt} are all positive, we can take logarithms of both sides of this equation to obtain:[35]

$$\log \omega_{ijt} = \log \mu_{ij} + (1 - \varepsilon_M) \log p_{ijt}. \tag{34}$$

Clearly, there will be some industry i that does not source any inputs from industry j, so $\mu_{ij} = 0$. This logarithmic transformation is thus applicable only to those nontrivial (i, j) pairs from which we would aim to identify ε_M. Given a sample of T observations, we have that:

$$\frac{\sum_{i=1}^{T} (\log \omega_{ijt} - \overline{\log \omega_{ij}})^2}{T-1} = (1 - \varepsilon_M)^2 \frac{\sum_{t=1}^{T} (\log p_{ijt} - \overline{\log p_{ij}})^2}{T-1}, \tag{35}$$

Denoting the two sample variances by $\hat{\sigma}^2(\log \omega_{ijt})$ and $\hat{\sigma}^2(\log p_{ijt})$, we have that the model implies that:

$$(1 - \varepsilon_M)^2 = \frac{\hat{\sigma}^2(\log \omega_{ijt})}{\hat{\sigma}^2(\log p_{ijt})}. \tag{36}$$

as long as $\hat{\sigma}^2(\log p_{ijt}) > 0$. Equation (36) implies that if the ratio of variances of (logs of) intermediate input shares and prices equals 1, $\varepsilon_M = 0$ which corresponds to a Leontief aggregation of intermediate inputs. If instead intermediate input shares do not vary at all (so $\hat{\sigma}^2(\log \omega_{ijt}) = 0$), then $\varepsilon_M = 1$, i.e. intermediates are aggregated in a Cobb-Douglas fashion. Finally, if instead the variability of intermediate input shares is relatively high compared to that of prices, then ε_M will be tend to be larger than 1 (assuming $\varepsilon_M > 0$), which corresponds to a case of highly substitutable inputs.

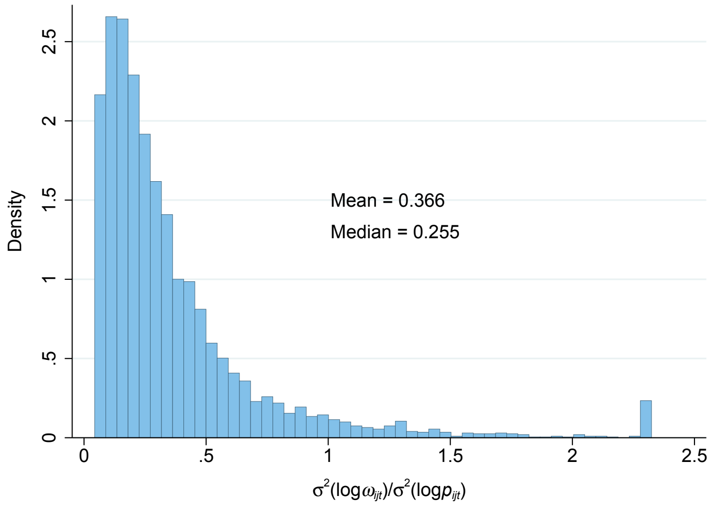
The relationship given by equation (36) is model-based, while its right-hand side is observable. Figure 8 therefore shows the empirical distribution of the ratio on the right-hand side of equation (36).[36] The distribution is right-skewed and has a mean (median) equal to around 0.37 (0.26). Assuming that $\varepsilon_M > 0$, this implies that the value of ε_M corresponding to the mean (median) of this distribution equals 0.40 (0.50).

We now estimate ε_M more formally. As in Atalay (2017), we use industries' first-order condition with respect to intermediate inputs, and exploit their heterogeneous (direct and indirect) exposures to military spending, which we use to construct the instrument that

[35] Although this equation comes from the simple model in Section 3, it is completely analogous to equation (41), assuming away time-effects (ϕ_t) and that the model is "true" ($\nu_{ijt} = 0$).

[36] We make use of the supply and use tables as well as data on industries' deflators; see Appendix A.

Figure 8. Empirical density of the ratio of sample variances of input shares and prices (see equation (36) in the main text)



Notes: The horizontal axis corresponds to the ratio of the sample variances of intermediate input shares and prices.

overcomes the endogeneity of prices in the first-order condition. The cost minimisation condition of the industry i representative firm gives the relationship between the share of intermediate inputs from industry j in industry i's total intermediate consumption:

$$\Delta \log \left(\frac{P_{jt} M_{ijt}}{P_{it}^M M_{it}} \right) = \phi_t + (1 - \varepsilon_M) \Delta \log \left(\frac{P_{jt}}{P_{it}^M} \right) + \nu_{ijt}. \tag{37}$$

Intuitively, assuming that military spending on industries' intermediate inputs is exogenous renders it as a valid instrument since it will be uncorrelated with the regression error term (ν_{ijt}).

Table 2 shows the results of our estimation using UK data. OLS yields estimates of ε_M around 0.68, though these estimates are inconsistent insofar as P_{jt}/P_{it}^M is endogenous in equation (37). IV estimation—which uses, in the first stage, three sets of instruments based on industries' heterogeneous exposures to UK military spending—yields estimates of ε_M between 0.27–0.35, depending on whether year fixed effects, ϕ_t, in equation (37) are included or not. All instruments are significant at the 1% level and the second-stage estimate of ε_M is significant at the 5% level in our preferred specification, allowing for year fixed effects. Note that the estimated value of ε_M of 0.35 in our preferred specification is very close to 0.4, which we obtained in a cruder fashion using equation (36). We thus set ε_M equal to 0.35. We calibrate the remaining elasticity parameters following Atalay (2017).[37]

[37] Unlike in the baseline calibration in Atalay (2017), the value of ε_Q estimated or calibrated in a few other recent papers has tended to be somewhat smaller than 1. For example, Peter and Ruane (2020) estimate a value of 0.6 for the US, and Baqaee and Farhi (2019) use 0.5. Our results are generally robust to an alternative calibration with $\varepsilon_Q = 0.5$. Our results are similarly robust to alternative (non-unitary)

Table 2: Regression results related to equation (37)

Second stage	(1) OLS	(2) OLS, Year FE	(3) IV	(4) IV, Year FE
ε_M	0.680***	0.680***	0.273***	0.345**
	(0.015)	(0.015)	(0.266)	(0.263)
First stage: Dependent variable is $\Delta \log P_{tI} - \log P_{tJ}^{in}$.				
military spending shock$_{tI}$			0.331***	0.385***
			(0.033)	(0.050)
military spending shock$_{tJ}$			-0.205***	-0.185***
			(0.029)	(0.032)
military spending shock$_{tJ\text{'s suppliers}}$			-0.051***	-0.045***
			(0.011)	(0.012)
N	35343	35343	34623	34623
Adjusted R^2	0.013	0.014	.	.
Wu-Hausman test p-value			0.124	0.204
Cragg-Donald Statistic			37.968	38.593
Year Fixed Effects	No	Yes	No	Yes

Notes: Statistical significance: * $p < 0.10$, ** $p < 0.05$, *** $p < 0.01$.

Regarding the returns to scale parameters η_i, our baseline calibration maintains the assumption of constant returns to scale.[38][39] Using UK supply and use tables, we calibrate α_i, ξ_i, μ_i, Γ_{ij}^M, and Γ_{ij}^X. We set the quarterly depreciation rate to equal 1.4% to be consistent with the annual depreciation rate of around 5.5%, which is roughly the average estimated by Oulton and Wallis (2016) since 1997.[40] The calibrated values of all parameters in the model are summarised in Table 3.

reasonable calibrations of ε_D.

[38]An alternative calibration we used involved estimating production function parameters for each industry using firm-level BvD data for the UK. Using that approach, we set η_i equal to the sum of resulting output elasticities with respect to inputs. Our results are largely unchanged across the baseline and this alternative calibration of the returns to scale parameters.

[39]As we note elsewhere in the paper—see equation (122)—the relationship between outdegrees and prices is independent of the returns to scale in both the (log-linearized) dynamic model and the simpler static model. But an important difference between the two models is that in the static model, preference shocks leave relative prices unaffected under constant returns to scale, which would leave the matrix $\mathbf{W}_t$ and so outdegrees (so producer centrality) unchanged. This is not the case in the dynamic model–even under constant returns to scale, preference shocks can affect relative prices (and so producer centrality).

[40]For simplicity, we assume that the (time-invariant) depreciation rate does not vary across sectors. In practice, there is likely to be both time-variation in the depreciation rate (even if the *asset-specific* depreciation rates are constant) as the composition of a sector's capital assets changes over time (e.g. due to greater use of IT, which tends to have higher depreciation than buildings, for example). Our findings are not sensitive to reasonable variations of the depreciation rate.

Table 3. Baseline calibration of model parameters

Parameter(s)	Value	Source
ε_Q	1	Atalay (2017)
ε_M	0.35	Estimated
ε_X	1	Atalay (2017)
ε_D	1	Atalay (2017)
ε_{LS}	2	Atalay (2017)
η_i	1	Atalay (2017)
α_i	Average share of labour expenses in J's GVA	UK's Supply and Use Tables
ξ_i	Average J's share of final demand	UK's Supply and Use Tables
μ_i	Average share of intermediates in J's GVA	UK's Supply and Use Tables
δ_K	0.014	Based on Oulton and Wallis (2016)
Γ^M_{ij}	Average share of intermediate inputs from j to i	UK's Supply and Use Tables
Γ^X_{ij}	Average share of GFCF flows from j to i	UK's Supply and Use Tables
β	0.99	Atalay et al. (2018)

Notes: 'Averages' refer to average values over 1997-2019. i and j denote 2-digit industries.

Having log-linearised and fully calibrated the model, we now describe our approach to filtering out all shocks in the model.

4.6 Model Filter

In a novel application, we use quarterly data on UK industries' value-added and prices to filter out the technology and preference shocks in the model. Denote the vector of log-deviations of value-added from its steady state as $\hat{v}_t$. Note that since we assume $\varepsilon_Q = 1$, we have that the log-deviations of value-added equal those of gross output, i.e. $\hat{v}_t = \hat{q}_t$ (see Appendix C.3). Therefore, the value-added data, which the model will match exactly, will also correspond to industries' real output in the model. We target value-added since we do not observe output growth by industry at a quarterly frequency. Recall also that the first-order outdegrees can be expressed as functions of prices only.[41] Therefore, by exactly matching the data on industries' prices, we ensure that the model matches industries' first-order outdegrees as well as possible.

Appendix C.3 shows that our model filter follows a VARMA(1,1) process:

$$\begin{bmatrix} \Delta\hat{v}_{t+1} \\ \Delta\hat{p}_{t+1} \end{bmatrix} = \begin{bmatrix} \tilde{\mathbf{V}}_v & \mathbf{0} \\ \mathbf{0} & \tilde{\mathbf{P}}_p \end{bmatrix} \begin{bmatrix} \Delta\hat{v}_t \\ \Delta\hat{p}_t \end{bmatrix} + \begin{bmatrix} \mathbf{V}_a & \mathbf{V}_d \\ \mathbf{P}_a & \mathbf{P}_d \end{bmatrix} \begin{bmatrix} \omega^A_{t+1} \\ \omega^D_{t+1} \end{bmatrix} + \begin{bmatrix} \tilde{\mathbf{V}}_a & \tilde{\mathbf{V}}_d \\ \tilde{\mathbf{P}}_a & \tilde{\mathbf{P}}_d \end{bmatrix} \begin{bmatrix} \omega^A_t \\ \omega^D_t \end{bmatrix}, \tag{38}$$

where the matrices in bold only depend on the model parameters listed in Table 3. By expressing each of the shocks as a function of data and the lagged shock itself and assuming that the initial shocks are zero, we can solve equation (38) forward to filter out both

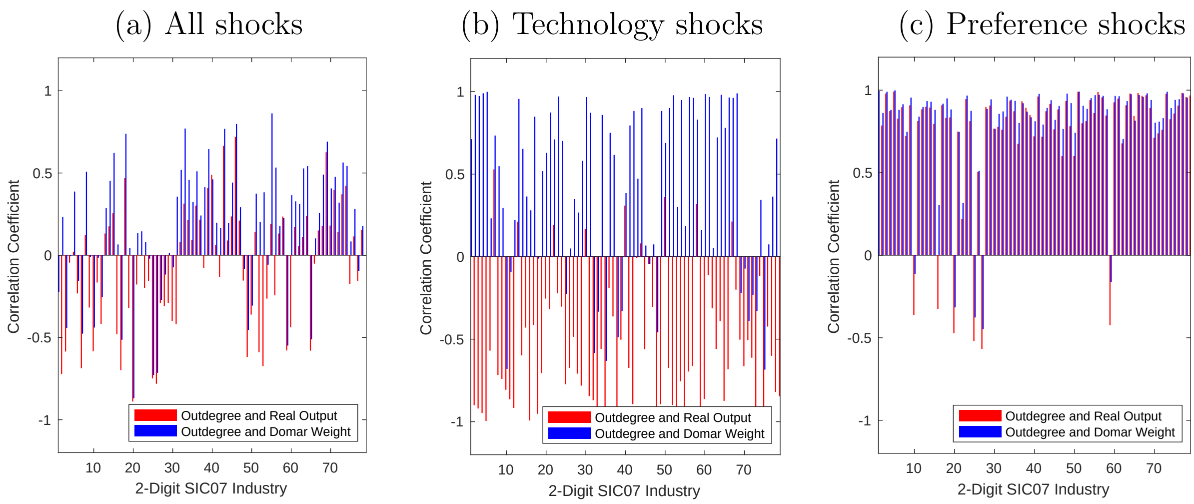
[41] In the model's log-linear form, this also applies to higher-order measures of centrality, such as Bonacich centrality.

technology and preference shocks.[42] Alternatively, we can use the Kalman filter. The two methods are generally equivalent.[43,44]

Once we have filtered out the shocks, we can back out the model-implied size and centrality variables—namely real gross output and Domar weights, and first-order weighted outdegrees—using equations (32) and (33), respectively.

Panel (a) in Figure 9 shows the resulting correlations between our measures of size and centrality implied by the filtered technology and preference shocks. The broad pattern across industries shows mostly positive correlations between steady-state deviations of size and centrality, similar to its empirical counterpart shown in Figure 7. The discrepancies between the empirical and model-implied size-centrality relationship arise mainly a consequence of (i) the fact that the model-implied relationship between outdegrees and prices $\hat{d}_t^{out} = \mathbf{D}\hat{p}_t$ does not hold exactly in the data, and (ii) the fact that value-added and output growth are not identical in the data, unlike in the model.

Figure 9. Model-implied relationship between size and centrality



Panel (b) in Figure 9 shows the implied size-centrality relationship resulting from the filtered technology shocks only. We see that technology shocks tend to induce positive correlation between Domar weights and outdegrees, but a negative one between real output and outdegrees, consistent with our findings from the static model shown in Table

[42]We set the burn-in period to 7 quarters, so we discard the first 7 values of each filtered shock.

[43]More specifically, the convergence of the model filter will depend on the eigenvalues of

$$\begin{bmatrix} \mathbf{V}_a & \mathbf{V}_d \\ \mathbf{P}_a & \mathbf{P}_d \end{bmatrix}^{-1} \begin{bmatrix} \tilde{\mathbf{V}}_a & \tilde{\mathbf{V}}_d \\ \tilde{\mathbf{P}}_a & \tilde{\mathbf{P}}_d \end{bmatrix},$$

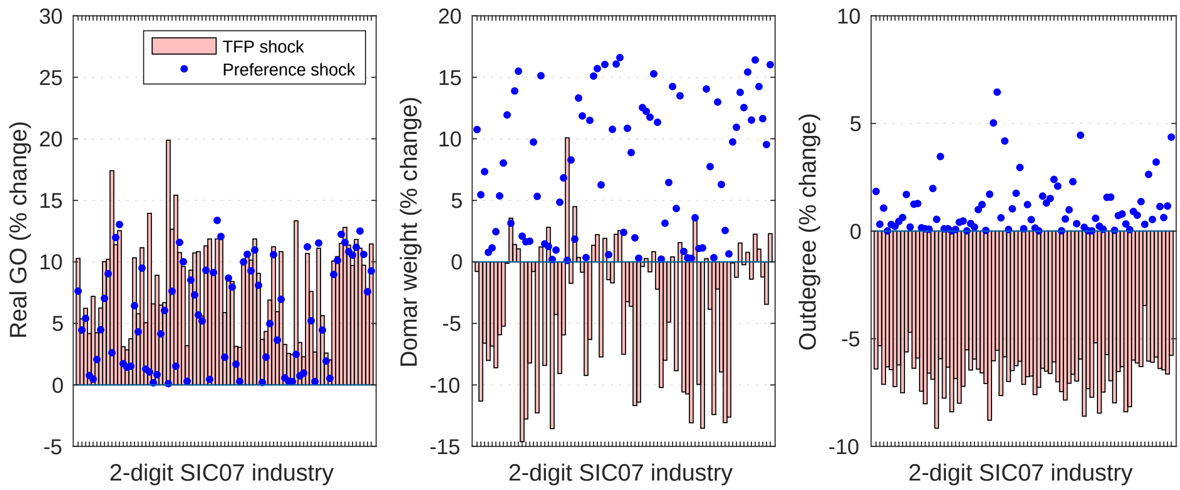
being less than 1 in modulus. See Atalay (2017) for a more elaborate discussion.

[44]Note that we do not need data on the state variable (capital, $\hat{k}_t$) to do this. As shown by Atalay (2017), after solving for a variable in terms of capital and the shocks, the first-order log-linear approximation to the equilibrium dynamics around the steady state has a VARMA representation, as long as an invertibility condition is satisfied.

1. Such strong negative co-movement between outdegrees and real output outside of the steady state is at odds with the data (Figure 7). In contrast, panel (c) in Figure 9 shows that preference shocks alone induce very strong positive correlation between either measure of producer size and centrality. Therefore, to match the empirical size-centrality relationship as shown in Figure 7, a combination of supply and demand-type shocks is needed. The relative importance of each type of shocks (which will partly depend on the targets in the filtering procedure) will determine the extent to which the model can match the empirical size-centrality relationship.

Although shocks transmit through the production network, the majority of variation in value-added and prices will result from industries' own shocks. To understand further the patterns in Figure 9, it is therefore instructive to focus on 'own-effects' of shocks. Figure 10 shows the contemporaneous responses of our size and centrality measures to a positive 10% technology (or preference) shock for each industry.[45,46] As in the simple model in Section 3, both technology and preference shocks that are positive raise own real output. Domar weights increase in the face of own preference shocks, and generally fall in response to own technology shocks. Finally, outdegrees strongly fall in response to technology shocks, and increase (though to a lesser extent) in response to a preference shock. As industries' own shocks account for the bulk of variation in own size and centrality, these patterns thus help explain the results shown in Figure 9.

Figure 10. Contemporaneous effect to a +10% own technology/preference shock



Recall that in the static model, the implied size-centrality relationship—as a function

[45] We focus on the contemporaneous effects of shocks since the dynamics of the model feature relatively low persistence. Introducing capital adjustment costs in the model, for instance, would allow for greater persistence in the dynamic responses to shocks.

[46] Although a 10% shock is reasonable given the size of filtered innovations, scaling is not our concern since we want to illustrate the qualitative effects of these shocks; the effects of shocks are linear in the size of the shock.

of supply and demand shocks—depended crucially on the value of ε_M. Specifically, as long as ε_M was less than 1, the static model suggested supply-side shocks would move real output and centrality in opposite directions, unlike preference shocks. By varying the value of ε_M and filtering out the resulting sets of shocks (keeping other model parameters the same as in our baseline calibration), we find that the same result holds true in the estimated dynamic model (see Table 4). With $\varepsilon_M > 1$, the effects of shocks on the size-centrality relationship generally flips sign. Although the size-centrality relationship in the data is not a targeted moment in the model filter, we nonetheless find that our baseline calibration matches it better than alternative values of ε_M shown in Table 4. In Table D.1 in Appendix D, we show that changing two other key parameters (ε_Q and ε_D) does not change our fundamental findings about the effects of shocks on size and centrality.

Table 4: Implied Size Centrality Relationship in Dynamic Model Under Different Values of ε_M vs. Empirical Data Counterpart for the UK

Implied Size-Centrality Relationship	Elasticity of Substitution Across Intermediates (ε_M)							Data
	0.1	0.2	0.35 (*baseline*)	0.5	0.8	1	1.5	
$\overline{\text{corr}}(\hat{d}_t^{out}, \hat{q}_t)$, all shocks	0.03	0.02	0.00	-0.05	-0.21	-0.14	-0.27	0.12
$\overline{\text{corr}}(\hat{d}_t^{out}, \hat{\lambda}_t)$, all shocks	0.31	0.27	0.23	0.13	0.06	0.11	-0.43	0.17
$\overline{\text{corr}}(\hat{d}_t^{out}, \hat{q}_t)$, only technology shocks	-0.49	-0.54	-0.67	-0.70	-0.77	-0.76	0.03	n.a.
$\overline{\text{corr}}(\hat{d}_t^{out}, \hat{\lambda}_t)$, only technology shocks	0.73	0.69	0.58	0.46	-0.04	-0.45	0.66	n.a.
$\overline{\text{corr}}(\hat{d}_t^{out}, \hat{q}_t)$, only preference shocks	0.89	0.88	0.86	0.84	0.81	0.75	-0.59	n.a.
$\overline{\text{corr}}(\hat{d}_t^{out}, \hat{\lambda}_t)$, only preference shocks	0.92	0.91	0.90	0.90	0.88	0.84	-0.81	n.a.

Notes: $\hat{d}_t^{out}$ denotes producer centrality, $\hat{q}_t$ denotes real gross output, and $\hat{\lambda}_t$ denotes Domar weight (all in terms of steady-state log deviations).

In summary, as in the simple model in Section 3, technology shocks tend to induce negative (positive) comovement between real output (Domar weights) and outdegrees, unlike preference shocks which tend to induce a positive size-centrality relationship. More importantly, we show that demand-type shocks are needed in order to reconcile the model-implied size-centrality relationship outside of steady state (Figure 9) with its empirical counterpart (Figure 7).

Finally, recall that we found that the empirical size-centrality relationship in steady state tends to be positive (Figure 6). This is also the case in this model, given our calibration approach. For example, the steady-state input-output shares (γ_{ij}) are calibrated to match the average empirical counterparts over our sample, which underpin the empirical results shown in Figure 6. The dynamic model is therefore able to match the empirical size-centrality relationship both in and out of steady state.

5 UK Productivity Growth Puzzle from a Production Network Perspective

5.1 Deriving the Decompositions

We now focus on analysing the productivity growth slowdown in the UK following the 2008–09 recession using the dynamic model. The purpose of this application is to (i) show that locating the growth 'puzzle' is different from identifying the underlying shocks, and (ii) investigate how the existence of the production network—in particular, the heterogeneity in industries' size and centrality—affects (i).

Defining aggregate labour productivity as aggregate value-added divided by aggregate labour, we can write it in log-linear form in the model from Section 4 as:

$$\hat{Y}_t - \hat{L}_t = \sum_{i=1}^{N} \left(S_i^Y \hat{v}_{it} - L_i \hat{l}_{it} \right) = S^Y \cdot \hat{v}_t - L \cdot \hat{l}_t, \tag{39}$$

where S_i^Y corresponds to the steady-state share of industry i's value-added in aggregate value-added:

$$S_i^Y = \frac{(1-\mu_i)P_i Q_i}{\sum_{i=1}^{N}(1-\mu_i)P_i Q_i}, \tag{40}$$

and L_i denotes industry i's steady-state share of aggregate labour. The growth of aggregate labour productivity is thus given by:

$$\begin{aligned} \Delta\hat{Y}_t - \Delta\hat{L}_t &= S^Y \Delta\hat{v}_t - L\Delta\hat{l}_t \\ &= S^Y \left(\mathbf{V}_k \Delta\hat{k}_t + \mathbf{V}_a \omega_t^A + \mathbf{V}_d \omega_t^D \right) - L \left(\mathbf{L}_k \Delta\hat{k}_t + \mathbf{L}_a \omega_t^A + \mathbf{L}_d \omega_t^D \right). \end{aligned} \tag{41}$$

where the matrices in bold depend on the model parameters only.[47] Equation (41) shows that we can decompose aggregate labour productivity growth into contributions from industries and/or contributions from shocks. More precisely, since value-added and labour are themselves VARMA(1,1) processes involving the underlying shocks, we can express aggregate labour productivity itself in terms of the shocks. In doing historical decompositions of aggregate labour productivity growth, our interest will be in distinguishing the contribution of an industry i (which could, in principle, reflect all shocks) from the contribution of idiosyncratic shocks to industry i (which may affect the aggregate by also transmitting to other industries).

In addition to idiosyncratic shocks, there may be common shocks that affect possibly

[47] Equation (41) makes use of equations (115) and (126) derived in Appendices C.3 and C.5, respectively.

all industries. Once we have filtered out the shocks, we perform factor analysis to extract the common component. In particular, we assume that there are possibly two common factors affecting industries' technology.[48] We assume away the existence of common preference shocks: unlike aggregate technology shocks, aggregate preference shocks do not have an intuitive interpretation, but industry-specific preference shocks do.[49]

The relative importance of common shocks versus industry-specific shocks will mainly depend on two factors: (i) the extent to which the targeted variables are correlated across industries (in our case, value-added and labour growth), and (ii) the value of ε_M. If ε_M is close to 0, industries source inputs from each other in an almost complementary fashion, implying they do not substitute across them easily. In that case, a shock to a highly central industry (such as finance) will act much like a common shock since many industries are exposed to it and cannot substitute away. If instead ε_M is close to (or larger than) one, industries can substitute across their input-suppliers relatively more easily so genuine common shocks will be a much more likely source of cross-correlation in the observed value-added/labour growth than shocks to any particular industry, no matter how central that industry may be.

Since we are considering a first-order approximation around a non-stochastic steady state, aggregate labour productivity depends on industries' value-added and labour through the time-invariant matrices, S^Y and L. In our baseline calibration, the values of the entries in these matrices are related to the average data counterparts over our sample (1997–2019).[50] As a robustness check, we also consider two alternative calibrations: one based on the 1997 data, and another based on the 2019 data. These alternative calibrations yield very similar results to our baseline findings (see Figure 14 in Appendix D).

Note that backing out the shocks by matching industries' value-added and prices growth (as in Subsection 4.6) would not allow us to capture the actual movements in industries' labour sufficiently well (as this would not be a data moment that is targeted). For this reason and for the purpose of this application, we back out the shocks by matching

[48]By construction, the common and sectoral (idiosyncratic) components add up to the total shock for each industry. Note that Atalay (2017), whose dataset contains 30 sectors, allows for a single common component. We have 79 industries so allowing for an additional common factor seems appropriate. This is also comparable to Foerster et al. (2011), who allow for two common factors in a model with 117 sectors.

[49]Nonetheless, allowing for common preference shocks, consisting of one or two common factors, has a negligible effect on our results.

[50]An important phenomenon that has taken place since the late 1990s in the UK has been the decline of manufacturing sector's share of labour, up until the 2008-09 crisis (from 16% in 1997 to 9% in 2009). The steady-state matrix of industries' shares of total labour, L, will thus not reflect the declining relative importance of manufacturing for aggregate productivity growth *solely* due to its share of total labour falling.

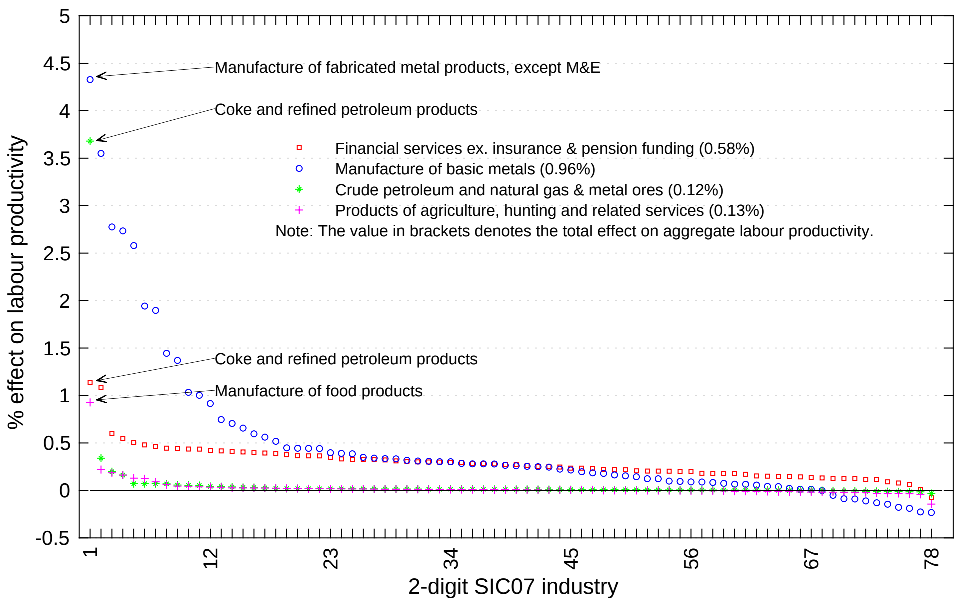
the industries' value-added and labour growth in the data using:

$$\begin{bmatrix} \Delta\hat{v}_{t+1} \\ \Delta\hat{l}_{t+1} \end{bmatrix} = \begin{bmatrix} \tilde{\mathbf{V}}_v & \mathbf{0} \\ \mathbf{0} & \tilde{\mathbf{L}}_l \end{bmatrix} \begin{bmatrix} \Delta\hat{v}_t \\ \Delta\hat{l}_t \end{bmatrix} + \begin{bmatrix} \mathbf{V}_a & \mathbf{V}_d \\ \mathbf{L}_a & \mathbf{L}_d \end{bmatrix} \begin{bmatrix} \omega^A_{t+1} \\ \omega^D_{t+1} \end{bmatrix} + \begin{bmatrix} \tilde{\mathbf{V}}_a & \tilde{\mathbf{V}}_d \\ \tilde{\mathbf{L}}_a & \tilde{\mathbf{L}}_d \end{bmatrix} \begin{bmatrix} \omega^A_t \\ \omega^D_t \end{bmatrix}. \tag{42}$$

Again, all of the matrices determining the dynamics of the model are functions of the model parameters listed in Table 3 only.

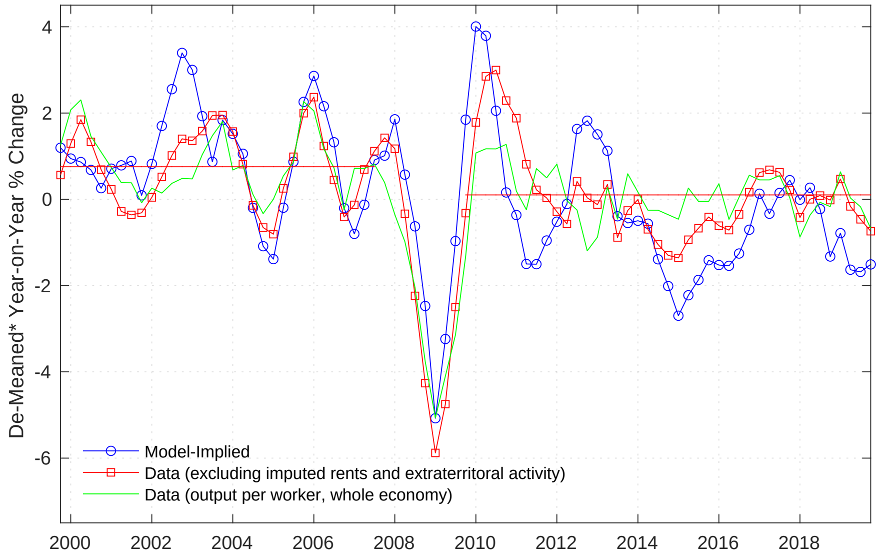
The attribution of the productivity growth slowdown to industries in equation (41) will be affected by their size and centrality in two ways. First, for a given $\Delta\hat{v}_t$ or $\Delta\hat{l}_t$, larger industries will affect aggregate labour productivity more to the extent that their share of total value-added, S^Y, and share of total labour, L, is larger. Second, column $j \in N$ in the matrices $\mathbf{V}_i$ ($\mathbf{L}_i$) for $i \in (k, a, d)$, which will determine how value-added (labour) in all industries respond to changes in industry j's capital and its technology and preference shocks, will generally have larger entries (in absolute size) for larger and/or more central industries. Figure 11 illustrates this visually: it shows the effect of a 10% technology shock in selected industries on *other* industries' labour productivity. We can see that shocks to the most central industries in the input-output network (e.g. basic metals and finance) affect a disproportionately large number of other industries relative to shocks to less central industries (e.g. agriculture and mining).

Figure 11. Ordered responses of other industries' labour productivity to a 10% technology shock in selected four industries



Notes: The x-axis has 78 ticks, corresponding to 78 industries other than the industry which is the source of the technology shock (in total, there are $N = 79$ industries). In each of the four cases, the first tick corresponds to the most affected industry other than the industry in which the shock has originated, the second tick to the second most affected industry other than the industry in which the shock has originated, and so on.

Figure 12. Aggregate labour productivity de-meaned growth: data vs. model



Notes: *The model-implied series is based on de-meaned data on industries' value-added and jobs growth. The two series based on ONS data have been de-meaned so as to ensure that all three series have the same mean over the period shown in the figure (1999:Q4 to 2019:Q4). The two horizontal lines show the pre-2008 and post-2010 averages in the data.

5.2 UK's Productivity Growth Puzzle

The UK experienced very steady and relatively strong productivity growth prior to the onset of the 2008–09 recession, with a clear slowdown of productivity growth post-crisis. Many authors have referred to this slowdown as the UK's productivity puzzle, or more precisely, the *growth puzzle*. Using equation (41), we can directly compute the model-implied aggregate labour productivity growth, since the value-added and labour dynamics are exactly matched in the data. Figure 12 compares the resulting path with the data. The correlation coefficient between the model-implied path and the data is 0.8 (0.6) if imputed rents and extraterritorial activity are excluded (included).[51] A convenient way to conceptualise the growth puzzle is to think of it as the difference between average post-crisis and pre-crisis growth. Throughout this section, we treat the period from 1999Q1–2007Q4 as 'pre-crisis', and 2010Q1–2019Q4 as 'post-crisis'. We find that the model-implied growth puzzle is around 0.26 percentage points (pp) lower growth per quarter post-crisis, somewhat larger but close to the data counterpart of 0.18pp.

Using the filtered shocks from equation (42) and the dynamics of aggregate labour productivity growth given by equation (41), we can analyse this slowdown by going beyond

[51] We exclude imputed rents and extraterritorial activity (section T) in the model, but we include the ONS's headline measure of output per worker for completeness. We do not exactly match the official ONS aggregate data for two reasons. First, the log-linear approximation of the model dynamics does not exactly match the ONS's sectoral aggregation. Second, the parameters of the matrices S^Y and L are time-invariant over the sample, unlike in the data.

the accounting decomposition of the slowdown into industries' contributions and analysing the importance of various shocks in driving the puzzle. We make this distinction clear below.

5.2.1 Accounting-Type Contributions of Industries

First, using equation (41), we can decompose aggregate labour productivity growth into industries' contributions. These contributions reflect all underlying shocks—common or industry-specific—that have impacted a given industry and thereby affected aggregate labour productivity. This exercise is conceptually similar to the decomposition in Tenreyro (2018), who finds that three quarters of the UK's productivity growth puzzle can be accounted for by manufacturing and finance.[52]

The top panel (a) of Figure 13 shows the results of our historical decomposition. We can see that the dynamics of aggregate productivity have been significantly driven by the manufacturing sector. Albeit significantly smaller, the contributions from finance are also non-negligible. Note that these contributions reflect potentially all underlying shocks, be it industry-specific or common. The difference between the average pre-crisis and post-crisis contribution to quarterly aggregate productivity growth from manufacturing is -0.37pp. Therefore, the post-2010 slowdown in manufacturing growth alone can account for more than the entirety of the aggregate growth puzzle (-0.26pp). Other sectors that have contributed negatively (albeit to a much lesser extent than manufacturing) include mining, finance, and ICT. By contrast, real estate along with admin and support and public services have provided a partial offset given their better productivity performance post-crisis than pre-crisis.

5.2.2 Contributions of Industry-Specific and Common Shocks

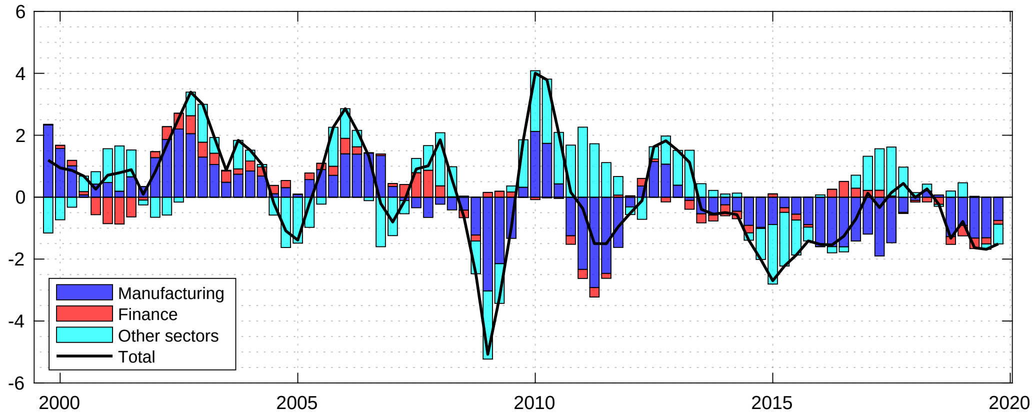
Using equation (41), we can also decompose aggregate labour productivity growth into the contributions from the underlying shocks, including any common shocks. Intuitively, the total contribution of the (filtered) idiosyncratic shock to, say, finance will include its effect on aggregate labour productivity via potentially all industries, not only finance. Panel (b) in Figure 13 shows the results.

We can see that the common technology shock has been important around the 2008-9 crisis years. On net, we find that its contributions have been generally more positive post-crisis than pre-crisis. By contrast, the contributions of manufacturing-specific shocks have been persistently negative and very sizeable since the crisis, in contrast to the

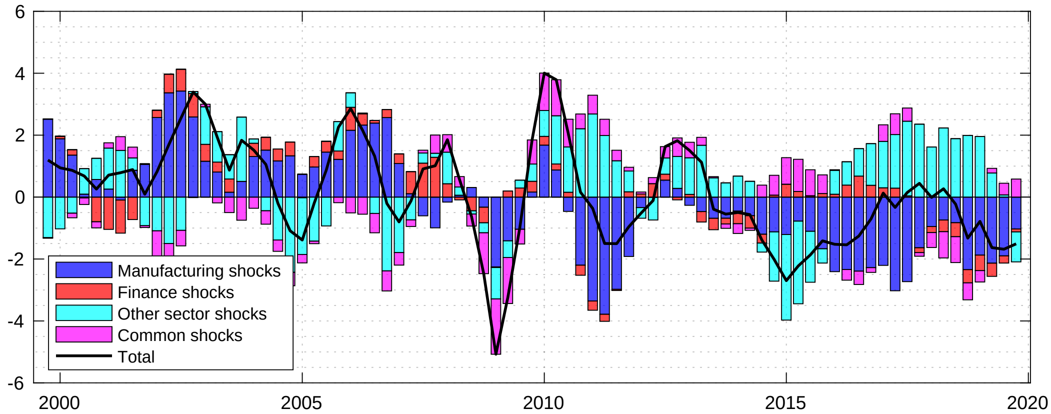
[52]The main difference relative to Tenreyro (2018) is that the linearity of aggregate labour productivity in sectoral contributions and/or shocks here comes from the log-linear nature of the approximation around the steady state. Instead, the approach in Tenreyro (2018) directly exploits the bottom-up aggregation of national statistics.

Figure 13. Historical decomposition of contributions to aggregate labour productivity fluctuations

(a) Accounting contributions (by sector)



(b) Shock contributions (by source of shock)



Notes: All 2-digit industries included.

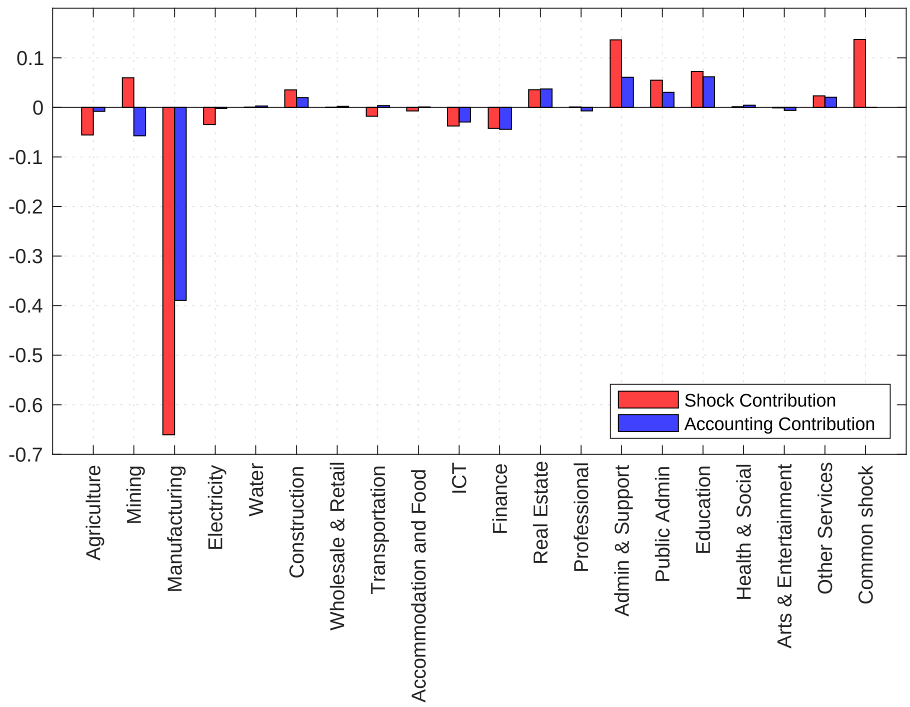
pre-crisis period. It also appears that the manufacturing sector's *accounting* contributions, shown in panel (a), tend to subsume the contributions from common technology shocks to a meaningful degree. Intuitively, although technology shocks in this model propagate both upstream (i.e. towards one's input suppliers) and downstream (i.e. towards one's input purchasers), the downstream propagation is stronger.[53] As the most central input-supplying sector in the economy by far, manufacturing thereby ends up absorbing most of the common technology shock.

We can compare the average contributions of idiosyncratic and common shocks post-crisis and pre-crisis; any differences in these contributions will have affected the growth puzzle. The red bars in Figure 14 show that the drag from more negative manufacturing-

[53]Intuitively, technology shocks change the prices faced by purchasers of inputs, creating powerful downstream propagation. See Acemoglu, Akcigit and Kerr (2015) for a more elaborate discussion of the propagation of shocks upstream and downstream in the network. Note that demand-side shocks have much more minor effects on prices and propagate mainly upstream as affected industries adjust their production levels and thus input demands.

specific shocks post-crisis has been particularly large, at -0.65pp per quarter. This has been partially offset by the common tehnology shocks which have been, on average, 0.13pp higher per quarter post-crisis than pre-crisis. Several sectors—most notably, administrative and support services activities ("Admin & Support") and mining and quarrying ("Mining")—have experienced significantly more positive shocks post-crisis relative to pre-crisis than their accounting contributions (reflecting possibly all shocks) would suggest.[54]

Figure 14. Contributions to the growth puzzle: sectors vs. shocks



In the previous section, we argued that the value of ε_M is key in determining the effect of shocks on the size-centrality relationship. Here, we find that our baseline results around the contributions of shocks to the UK's productivity growth puzzle are largely robust to varying ε_M (Table 5). The exceptions are that we find that common and finance-specific shocks generally tend to become more important in explaining the puzzle as ε_M increases. This is to be expected, as when ε_M is higher, the model filter is more likely to attribute common variation in industries' productivity dynamics to common factors (as it is easier to substitute away from idiosyncratic sources of variation). Importantly, as the most central sector in the economy, shocks to the financial sector act much like common shocks

[54]Another way to visualise the growth puzzle is to compare the performance of sectors post-crisis *over time* (i.e. without averaging) relative to their pre-crisis average performance. Figure D.1 in Appendix D shows that manufacturing is a clear outlier, having underperformed by far the most its pre-crisis growth, followed by finance, ICT and mining. Our results suggest fairly persistent deviations from pre-crisis contributions, hence the results echo those obtained by averaging pre- and post-crisis performance in Figure 14.

(i.e., it is difficult to substitute away from them, even with a high ε_M). That said, the ability of the model to match the aggregate productivity growth in the data generally falls meaningfully as ε_M rises above 0.5. In Table D.2 in Appendix D, we show that changing two other key parameters (ε_Q and ε_D) does not change our fundamental findings about the drivers of the UK's productivity growth slowdown.

Table 5: Contributions of Sector-Specific/Common Shocks to UK's Post-2010 Productivity Growth Puzzle for Different Values of ε_M

Contribution to Productivity Growth Puzzle	Elasticity of Substitution Across Intermediates (ε_M) 0.1	0.2	0.35 (*baseline*)	0.5	0.8	1	1.3	Data
Manufacturing-specific shocks	-0.63	-0.65	-0.65	-0.60	-0.66	-0.57	-0.31	n.a.
Finance-specific shocks	-0.03	-0.03	-0.04	-0.07	-0.08	-0.11	-0.16	n.a.
Other sectors' specific shocks	0.30	0.30	0.30	0.41	0.35	0.55	0.23	n.a.
Common shocks	0.12	0.13	0.13	-0.01	0.10	-0.13	-0.22	n.a.
Total growth puzzle	-0.24	-0.24	-0.26	-0.27	-0.28	-0.27	-0.46	-0.18
Correlation of model-implied aggregate productivity growth and data counterpart	0.79	0.78	0.77	0.75	0.70	0.62	0.25	n.a.

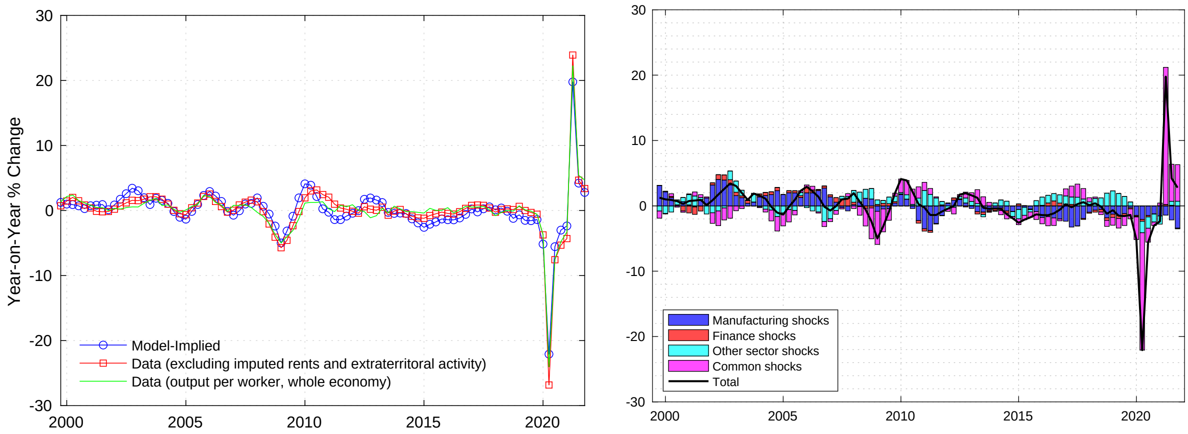
Notes: All other model parameters set according to the baseline calibration, shown in Table 3.

5.3 The Covid-2019 Pandemic

Our sample in the preceding analysis ended in 2019Q4, just before the Covid-19 pandemic. Including the subsequent two years in our sample[55], we find that the model continues to match the actual data well (left-panel in Figure 15). Looking at the contributions of shocks, our model suggests that the initial sharp downturn in 2020 as well as the subsequent jump in the year-over-year growth rate of aggregate productivity are primarily attributable to a common shock. This result is intuitive given the nature of the underlying pandemic shock, which entailed broad-based restrictions on social and economic activity.

[55]Given the unusual nature and size of the Covid-19 shock, we refrain from using the data in the period contaminated by the pandemic in our baseline analysis above, and the results in this section should be seen as illustrative.

Figure 15. Contributions to the growth puzzle: sectors vs. sectoral shocks (dashed)



Overall, casting the UK productivity growth puzzle into a multisector model with a production network allowed us to establish novel findings about its underlying drivers. First, the majority of variation in aggregate productivity growth is driven by industry-specific shocks. Second, the common technology shock has been more important around the crisis years and the Covid-19 pandemic, and its contributions have been generally more positive post-crisis than pre-crisis (excluding the Covid-19 period). Third, both in an accounting sense and as a source of industry-specific shocks, manufacturing has been by far the largest negative contributor to the UK's post-2010 slowdown in productivity growth. Finally, several sectors—most notably, admin and support services activities as well as a number of public services—have experienced notably more positive shocks post-crisis relative to pre-crisis than their accounting contributions (reflecting possibly all shocks) would suggest.

6 Conclusion

In this paper, we focused on two key characteristics of producers in a production network—their size and centrality—and their relationship in and outside of steady state. Existing literature shows that these characteristics are intimately related to the extent of shock transmission in production networks, both at a macro and micro level.

We analysed the characteristics of the UK economy's input-output network, showing that there are significant asymmetries in the degree of importance of industries as input suppliers. We showed that there is significant time-variation in the input-output network over time, which is inconsistent with a Cobb-Douglas aggregation of intermediate inputs in the production function. We also show that the empirical size-centrality relationship tends to be positive in steady state as well as away from it for most industries.

To conceptualise time-variation in the input-output network in a model with a production network, we set up a simple multisector model with a CES-type aggregation of intermediate inputs, featuring technology and demand shocks. We show that the link between shocks and producer centrality—and thus on its relationship with producer size—is closely related to the degree of input substitutability. As long as the elasticity of substitution is less than unitary, we show that technology shocks tend to induce negative (positive) co-movement between real output (Domar weights) and outdegrees, unlike preference shocks which tend to induce a positive size-centrality relationship.

We then consider a more general, dynamic model with an input-output network that we calibrate using UK data. We use this model to filter out technology and demand shocks, and show that the implied size-centrality relationship based on the filtered shocks can be reconciled with this empirical counterpart only in presence of both supply-side and demand-type shocks. These results thus confirm the conclusions reached within the simple model.

Finally, we use this model to analyse the UK's post-2010 productivity growth slowdown from a production network perspective, distinguishing industries' accounting contributions from the contributions of industry-specific and common shocks. We find that idiosyncratic shocks to the manufacturing sector have played a key role in driving the aggregate productivity slowdown.

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Appendices

A. Data

All data used in this paper is provided by the Office for National Statistics (ONS) and is publicly available.

Supply and Use Tables

The supply and use tables are published annually and available to download via this link. The vintage used in this paper was released on 29 October 2021.

Gross Value-Added (GVA) data

The data on industries' value-added can be obtained from the ONS's GDP(O) low-level aggregates dataset, available at this link. The vintage we use was released on 31 March 2022. The quarterly CVM GVA data are available from 1990Q1 to 2021Q4.

Jobs data

The data on jobs per industry can be downloaded via this link. The vintage we use was released on 15 March 2022. The quarterly jobs data are available from 1978Q2 to 2021Q4.

Deflators data

The data on industries' price deflators can be obtained via this link. The vintage we use was released on 15 March 2022.

Therefore, the full dataset used to obtain our baseline results and to filter out the shocks has $T = 87$ (1997Q2–2019Q4) and $N = 91$. The burn-in period is set to equal 7 quarters.

Seasonal adjustment

To seasonally adjust the data, we use the Census X-13 method in EViews.

B. Derivations Related to Section 3

B.1 Competitive Equilibrium

A *competitive equilibrium* of this economy consists of prices $(P_{1t}, ..., P_{Nt})$, wage W_t, labour $(L_{1t}, ..., L_{Nt})$, consumption $(C_{1t}, ..., C_{Nt})$, and quantities $(\{M_{ijt}\}_{i,j\in(1,...,N)})$ such that the representative consumer maximises her utility (subject to the budget constraint), the representative firms in each industry maximise profits, markets clear and the resource constraints are satisfied, that is

$$C_{it} + \sum_{j=1}^{N} M_{ijt} = Q_{it}, \qquad \forall i = 1, ..., N, \tag{43}$$

$$\sum_{i=1}^{N} P_{it} C_{it} = W_t L_t. \tag{44}$$

$$\sum_{i=1}^{N} L_{it} = L_t. \tag{45}$$

The consumer's problem is

$$\max_{\{C_{it}\}_{i\in(1,...,N)}} \prod_{i=1}^{N} C_{it}^{\gamma_{it}} - \frac{L_t^{\phi}}{\phi} \qquad \text{s.t.} \qquad \sum_{i=1}^{N} P_{it} C_{it} = W_t L_t. \tag{46}$$

The first-order condition (FOC) with respect to C_{it} is given by

$$\begin{aligned} \gamma_{it} C_{it}^{\gamma_{it}-1} \prod_{j\neq i}^{N} C_{jt}^{\gamma_{jt}} &= \mu_{it} P_{it}, \\ \frac{\gamma_{it} C_t}{C_{it}} &= \mu_{it} P_{it}. \end{aligned} \tag{47}$$

where μ_{it} is the Lagrange multiplier on the constraint. The FOC w.r.t. L_t is given by

$$L_t^{\phi-1} = \mu_{it} W_t. \tag{48}$$

We thus must have that:

$$\frac{\gamma_{it} C_t}{C_{it}} = \frac{L_t^{\phi-1}}{W_t} P_{it}. \tag{49}$$

Note that the ideal price index is given by $\mathrm{P}_t = \prod_{i=1}^{N} \left(\frac{P_{it}}{\gamma_{it}}\right)^{\gamma_{it}}$, which yields:

$$\frac{\gamma_{it} W_t L_t}{\mathrm{P}_t C_{it}} = \frac{L_t^{\phi-1}}{W_t} P_{it}. \tag{50}$$

Solving for C_{it}, we have that:

$$C_{it} = \frac{\gamma_{it} W_t^2 L_t^{2-\phi}}{P_{it}\mathrm{P}_t}. \tag{51}$$

To produce a given amount Q_{it}, the representative firm in each industry i faces the following cost-minimisation problem:

$$\min_{L_{it},\{M_{ijt}\}_{j\neq i\in(1,\ldots,N)}} W_t L_{it} + \sum_{j=1}^{N} P_{jt} M_{ijt} - \lambda_{it}\left[\left(Z_{it} L_{it}^{\alpha_i} M_{it}^{1-\alpha_i}\right)^{\eta_i} - Q_{it}\right].$$

The FOC with respect to L_{it} is given by:

$$W_t - \lambda_{it}\left(\eta_i \alpha_i L_{it}^{\eta_i\alpha_i - 1} Z_{it}^{\eta_i} M_{it}^{\eta_i(1-\alpha_i)}\right) = 0. \tag{52}$$

Multiplying both sides by L_{it}, we have that:

$$W_t L_{it} = \lambda_{it}\eta\alpha_i Q_{it}, \qquad \text{i.e.} \qquad L_{it} = \frac{\lambda_{it}\eta\alpha_i Q_{it}}{W_t}. \tag{53}$$

where λ_{it} is the Lagrange multiplier, and the FOC with respect to M_{ijt} is given by:

$$P_{jt} - \lambda_{it}\left(Z_{it}^{\eta_i} L_{it}^{\eta_i\alpha_i} \frac{\eta_i(1-\alpha_i)\varepsilon_M}{\varepsilon_M - 1} M_{it}^{\frac{\eta_i(1-\alpha_i)\varepsilon_M - \varepsilon_M + 1}{\varepsilon_M}} \mu_{ij}^{\frac{1}{\varepsilon_M}} \frac{\varepsilon_M - 1}{\varepsilon_M} M_{ijt}^{\frac{-1}{\varepsilon_M}}\right) = 0, \tag{54}$$

which can be simplified to obtain:

$$P_{jt} - \lambda_{it}\left(Z_{it}^{\eta_i} L_{it}^{\eta_i\alpha_i} \eta_i(1-\alpha_i) M_{it}^{\frac{\eta_i(1-\alpha_i)\varepsilon_M - \varepsilon_M + 1}{\varepsilon_M}} \mu_{ij}^{\frac{1}{\varepsilon_M}} M_{ijt}^{\frac{-1}{\varepsilon_M}}\right) = 0. \tag{55}$$

Multiplying both sides of this equation by M_{ijt} and rearranging, we have that:

$$M_{ijt} = \left(\frac{\lambda_{it}\eta_i(1-\alpha_i) Q_{it} M_{it}^{\frac{1-\varepsilon_M}{\varepsilon_M}} \mu_{ij}^{\frac{1}{\varepsilon_M}}}{P_{jt}}\right)^{\varepsilon_M}. \tag{56}$$

Let the natural price index for M_{it} be denoted by P_{it}^M. Since M_{it} is a CES aggregator, we have that the corresponding natural price index is given by:

$$P_{it}^M = \left[\sum_{j=1}^N \mu_{ij} P_{jt}^{1-\varepsilon_M}\right]^{\frac{1}{1-\varepsilon_M}}. \tag{57}$$

Using the FOC with respect to M_{ijt}, it then follows that:

$$P_{it}^M = \lambda_{it}\eta_i(1-\alpha_i)Q_{it}M_{it}^{-1}, \tag{58}$$

or, equivalently:

$$M_{it} = \frac{\lambda_{it}\eta_i(1-\alpha_i)Q_{it}}{P_{it}^M}. \tag{59}$$

By plugging the cost-minimising values of L_{it} and M_{it} back into the production function, we have that at the cost-minimising solution:

$$Q_{it} = Z_{it}^{\eta_i}\left(\frac{\lambda_{it}\eta_i\alpha_i Q_{it}}{W_t}\right)^{\eta_i\alpha_i}\left(\frac{\lambda_{it}\eta_i(1-\alpha_i)Q_{it}}{P_{it}^M}\right)^{\eta_i(1-\alpha_i)}. \tag{60}$$

We can then solve for λ_{it}, which yields:

$$\lambda_{it} = \zeta_i \frac{Q_{it}^{\frac{1-\eta_i}{\eta_i}}}{Z_{it}} W_t^{\alpha_i}(P_{it}^M)^{1-\alpha_i}, \tag{61}$$

where $\zeta_i = (\eta_i\alpha_i)^{-\alpha_i}\left[\eta_i(1-\alpha_i)\right]^{-(1-\alpha_i)}$. It then follows that the cost-minimising solutions for L_{it}, M_{it}, and M_{ijt} are given by:

$$L_{it} = \zeta_i\eta_i\alpha_i\frac{Q_{it}^{\frac{1}{\eta_i}}}{Z_{it}}W_t^{\alpha_i-1}(P_{it}^M)^{1-\alpha_i}, \tag{62}$$

$$M_{it} = \zeta_i\eta_i(1-\alpha_i)\frac{Q_{it}^{\frac{1}{\eta_i}}}{Z_{it}}W_t^{\alpha_i}(P_{it}^M)^{-\alpha_i}, \tag{63}$$

$$M_{ijt} = \frac{\zeta_i\eta_i(1-\alpha_i)W_t^{\alpha_i}(P_{it}^M)^{\varepsilon_i-\alpha_i}Q_{it}^{\frac{1}{\eta_i}}\mu_{ij}}{Z_{it}P_{jt}^{\varepsilon_M}} \tag{64}$$

Therefore, it follows that the total cost at the optimum is given by:

$$\begin{aligned} C(Q_{it}) &= W_t L_{it} + P_{it}^M M_{it} \\ &= Q_{it}^{\frac{1}{\eta_i}}\left(\frac{\zeta_i\eta_i\alpha_i}{Z_{it}}W_t^{\alpha_i}(P_{it}^M)^{1-\alpha_i} + \frac{\zeta_i\eta_i(1-\alpha_i)W_t^{\alpha_i}(P_{it}^M)^{\varepsilon_M-\alpha_i}}{Z_{it}}\sum_{j=1}^N \mu_{ij}P_{jt}^{1-\varepsilon_M}\right). \end{aligned} \tag{65}$$

Recalling that $\zeta_i = (\eta_i \alpha_i)^{-\alpha_i} [\eta_i(1-\alpha_i)]^{-(1-\alpha_i)}$, the marginal cost, $C'(Q_{it})$, is given by:

$$C'(Q_{it}) = \frac{Q_{it}^{\frac{1-\eta_i}{\eta_i}}}{Z_{it}} \left(\frac{W_t}{\eta_i \alpha_i}\right)^{\alpha_i} \left(\frac{P_{it}^M}{\eta_i(1-\alpha_i)}\right)^{1-\alpha_i}. \tag{66}$$

Since all markets are perfectly competitive, it must be the case that $P_{it} = C'(Q_{it})$, i.e.:

$$P_{it} = \frac{Q_{it}^{\frac{1-\eta_i}{\eta_i}}}{Z_{it}} \left(\frac{W_t}{\eta_i \alpha_i}\right)^{\alpha_i} \left(\frac{P_{it}^M}{\eta_i(1-\alpha_i)}\right)^{1-\alpha_i}. \tag{67}$$

Plugging the solutions for C_{it} (from the representative consumer's problem) and M_{jit} (from the representative firms' problem) into the market clearing condition, we have that:

$$Q_{it} = \frac{\gamma_{it} W_t^2 L_t^{2-\phi}}{P_{it} \mathrm{P}_t} + \sum_{j=1}^{N} \frac{\zeta_j \eta_j (1-\alpha_j) W_t^{\alpha_j} (P_{jt}^M)^{\varepsilon_M - \alpha_j} Q_{jt}^{\frac{1}{\eta_j}} \mu_{ji}}{Z_{jt} P_{it}^{\varepsilon_M}}. \tag{68}$$

Therefore, the equilibrium of this economy is fully characterised by a system of $3N+1$ equations in the same number of unknowns $((P_{it}))_{i=1,\dots,N}, (Q_{it}))_{i=1,\dots,N}, (L_{it}))_{i=1,\dots,N}, W_t)$ given by:

$$P_{it} = \frac{Q_{it}^{\frac{1-\eta_i}{\eta_i}}}{Z_{it}} \left(\frac{W_t}{\eta_i \alpha_i}\right)^{\alpha_i} \left(\frac{P_{it}^M}{\eta_i(1-\alpha_i)}\right)^{1-\alpha_i}, \tag{69}$$

$$Q_{it} = \frac{\gamma_{it} W_t^2 L_t^{2-\phi}}{P_{it} \mathrm{P}_t} + \sum_{j=1}^{N} \frac{\zeta_j \eta_j (1-\alpha_j) W_t^{\alpha_j} (P_{jt}^M)^{\varepsilon_M - \alpha_j} Q_{jt}^{\frac{1}{\eta_j}} \mu_{ji}}{Z_{jt} P_{it}^{\varepsilon_M}}, \tag{70}$$

$$L_{it} = \zeta_i \eta_i \alpha_i \frac{Q_{it}^{\frac{1}{\eta_i}}}{Z_{it}} W_t^{\alpha_i - 1} (P_{it}^M)^{1-\alpha_i}, \tag{71}$$

$$W_t \sum_{i=1}^{N} L_{it} = \prod_{i=1}^{N} C_{it}^{\gamma_{it}} \mathrm{P}_t. \tag{72}$$

where, without loss of generality, we take the consumer price index as the numeraire and set $\mathrm{P}_t = 1$.

B.2 Equilibrium intermediate input shares

By definition, the share of industry i's intermediate input expenses attributable to goods sourced from industry j is given by:

$$\omega_{ijt} \equiv \frac{P_{jt} M_{ijt}}{P_{it}^M M_{it}}. \tag{73}$$

Having solved for the optimal intermediate input demands in Appendix B.1, we have that the equilibrium intermediate input shares are given by:

$$\omega_{ijt} = \frac{P_{jt}\left(\frac{\zeta_i\eta_i(1-\alpha_i)W_t^{\alpha_i}(P_{it}^M)^{\varepsilon_M-\alpha_i}Q_{it}^{\frac{1}{\eta_i}}\mu_{ij}}{Z_{it}P_{jt}^{\varepsilon_M}}\right)}{P_{it}^M\left(\zeta_i\eta_i(1-\alpha_i)\frac{Q_{it}^{\frac{1}{\eta_i}}}{Z_{it}}W_t^{\alpha_i}(P_{it}^M)^{-\alpha_i}\right)}$$

$$= \mu_{ij}\left(\frac{P_{jt}}{P_{it}^M}\right)^{1-\varepsilon_M}. \tag{74}$$

B.3 First-Order and Second-Order Weighted Outdegrees

The first-order weighted outdegree of industry j is given by:

$$D_{jt}^{1,out} \equiv \sum_{i=1}^{N}\omega_{ijt} = \sum_{i=1}^{N}\mu_{ij}\left(\frac{P_{jt}}{P_{it}^M}\right)^{1-\varepsilon_M}. \tag{75}$$

By definition, the second-order weighted outdegree of industry j is given by:

$$D_{jt}^{2,out} \equiv \sum_{i=1}^{N}\omega_{ijt}D_{it}^{1,out}. \tag{76}$$

We now turn to characterising the responses of first-order and second-order weighted outdegrees in response to technology and demand shocks. Note that, by definition:

$$\frac{\partial D_{jt}^{2,out}}{\partial P_{jt}} = \sum_{i=1}^{N}\left[\frac{\partial\omega_{ijt}}{\partial P_{jt}}D_{it}^{1,out} + \omega_{ijt}\frac{\partial D_{it}^{1,out}}{\partial P_{jt}}\right],$$

Since P_{jt} has a weight of at most one in P_{it}^M, the first partial derivative $(\partial\omega_{ijt}/\partial P_{jt})$ is non-negative, and it will generally be positive as long as industry i sources inputs from industries other than industry j as well. Regarding the second partial derivative, note that:

$$\frac{\partial D_{it}^{out}}{\partial P_{jt}} = \sum_{k=1}^{N}\left[\mu_{ki}P_{it}^{1-\varepsilon_M}(\varepsilon_M-1)(P_{kt}^M)^{\varepsilon_M-2}\mu_{kj}P_{jt}^{-\varepsilon_M}(P_{kt}^M)^{\varepsilon_M}\right],$$

which is clearly negative if $\varepsilon_M < 1$. Note that the above equation can be rewritten as:

$$\frac{\partial D_{it}^{out}}{\partial P_{jt}} = \sum_{k=1}^{N}\left[\omega_{kit}(\varepsilon_M-1)\omega_{kjt}P_{jt}^{-1}\right].$$

If $i = j$, then:

$$\frac{\partial D_{jt}^{out}}{\partial P_{jt}} = \sum_{i=1}^{N} (1 - \varepsilon_M) \omega_{ijt} (1 - \omega_{ijt}) P_{jt}^{-1}.$$

Therefore, we have that the partial derivative of industry j's second-order weighted outdegree with respect to its price is given by:

$$\begin{aligned}
\frac{\partial D_{jt}^{2,out}}{\partial P_{jt}} &= \sum_{i=1}^{N} \frac{\partial \omega_{ijt}}{\partial P_{jt}} D_{it}^{1,out} + \sum_{i=1}^{N} \omega_{ijt} \frac{\partial D_{it}^{1,out}}{\partial P_{jt}} \\
&= \sum_{i=1}^{N} (1 - \varepsilon_M) \omega_{ijt} (1 - \omega_{ijt}) P_{jt}^{-1} \sum_{k=1}^{N} \omega_{kit} + \sum_{i=1, i \neq j}^{N} \omega_{ijt} \sum_{k=1}^{N} \left[\omega_{kit} (\varepsilon_M - 1) \omega_{kjt} P_{jt}^{-1} \right] \\
&+ \omega_{jjt} \sum_{i=1}^{N} (1 - \varepsilon_M) \omega_{ijt} (1 - \omega_{ijt}) P_{jt}^{-1} \\
&= (1 - \varepsilon_M) P_{jt}^{-1} \sum_{i=1, i \neq j}^{N} \omega_{ijt} \sum_{k=1}^{N} \omega_{kit} \left(1 - \omega_{ijt} - \omega_{kjt} \right) \\
&+ (1 - \varepsilon_M) P_{jt}^{-1} \omega_{jjt} \sum_{k=1}^{N} \omega_{kjt} \left(2 - \omega_{jjt} - \omega_{kjt} \right),
\end{aligned}$$

which can be rewritten as:

$$\frac{\partial D_{jt}^{2,out}}{\partial P_{jt}} = (1 - \varepsilon_M) P_{jt}^{-1} \left[\sum_{i=1}^{N} \omega_{ijt} \sum_{k=1}^{N} \omega_{kit} \left(1 - \omega_{ijt} - \omega_{kjt} \right) + \omega_{jjt} \sum_{k=1}^{N} \omega_{kjt} \right].$$

This derivative will be non-negative if:

$$\begin{aligned}
&\sum_{i=1}^{N} \omega_{ijt} \sum_{k=1}^{N} \omega_{kit} \left(1 - \omega_{ijt} - \omega_{kjt} \right) + \sum_{i=1}^{N} \omega_{ijt} \omega_{jjt} \geq 0, \qquad \text{i.e.} \\
&\sum_{i=1}^{N} \omega_{ijt} \omega_{jjt} \geq \sum_{i=1}^{N} \omega_{ijt} \sum_{k=1}^{N} \omega_{kit} \left(\omega_{ijt} + \omega_{kjt} - 1 \right).
\end{aligned}$$

The above condition can be written as:

$$D_{jt}^{2,out} + \omega_{jjt} D_{jt}^{1,out} \geq \sum_{i=1}^{N} \omega_{ijt} \sum_{k=1}^{N} \omega_{kit} \left(\omega_{ijt} + \omega_{kjt} \right), \tag{77}$$

which is strictly satisfied if $\omega_{ijt} + \omega_{kjt} < 1$, which will typically be the case. Note that if $\omega_{ijt} + \omega_{kjt} = 1$, then the right-hand side equals $D_{jt}^{2,out}$.

Note that the maximum value of $\omega_{ijt} + \omega_{kjt}$ for any $(i, k) \in N$ pair is 2. For a given ω_{kjt}, the maximum value of ω_{kit} is $1 - \omega_{kjt}$. So the maximum value of the right-hand side in equation (77) is given by the right-hand side of the inequality below:

$$\sum_{i=1}^{N} \omega_{ijt}\omega_{jjt} > \sum_{i=1}^{N} \omega_{ijt} \sum_{k=1}^{N} (1-\omega_{kjt})(\omega_{ijt}+\omega_{kjt}-1)$$

Suppose that $\omega_{jjt} = 0$, which makes the left-hand side of the above inequality as small as possible. For the inequality to still hold, we thus need that:

$$\sum_{i=1}^{N} \omega_{ijt} \sum_{k=1,k\neq j}^{N} (1-\omega_{kjt})(\omega_{ijt}-(1-\omega_{kjt})) + \sum_{i=1}^{N} \omega_{ijt}(1-\omega_{jjt})(\omega_{ijt}-(1-\omega_{jjt})) < 0$$

or, equivalently, that:

$$\sum_{i=1}^{N} \omega_{ijt} \sum_{k=1,k\neq j}^{N} \left[(1-\omega_{kjt})(\omega_{ijt}-\omega_{ijt}(1-\omega_{kjt})) - (1-\omega_{ijt})\right] < 0. \tag{78}$$

Treating ω_{ijt} as given, the expression inside the second sum is maximised when:

$$1-\omega_{kjt}-\omega_{ijt}-\omega_{kjt}+1 = 0, \qquad \text{i.e.} \qquad \omega_{kjt} = \frac{2-\omega_{ijt}}{2}.$$

If $k = i$, then clearly $\omega_{kjt} = 2/3$. It can then be easily shown that the term inside the second sum in equation (78) is given by:

$$(1-\omega_{kjt})(\omega_{ijt}-(1-\omega_{kjt})) - (1-\omega_{ijt}) = -\frac{3}{4}\omega_{ijt}^2 - \omega_{ijt} < 0 \quad \text{if} \quad \omega_{ijt} \neq 0.$$

Intuitively, since ω_{ijt} increases with P_{jt} for all i, then at least some of the remaining shares ω_{ikt} $(k \neq j)$ have to fall. This reduces other industries' $(k \neq j)$ outdegrees, but the sum of all outdegrees must always equal N, by definition. For industry j, the increases in ω_{ijt} more than offset the falls in $D_{it}^{1,out}$, so $D_{jt}^{2,out}$ is increasing in P_{jt}.

B.4 Derivation of equation (20)

Recall from equation (19) that industry j's Domar weight is given by:

$$\lambda_{jt} = \frac{P_{jt}Q_{jt}}{\mathrm{P}_tC_t} = \frac{P_{jt}C_{jt}}{\mathrm{P}_tC_t} + \sum_{i=1}^{N} \frac{P_{jt}M_{ijt}}{\mathrm{P}_tC_t}.$$

Using the definition of ω_{ijt}, we can rewrite the above equation as:

$$\lambda_{jt} = \frac{P_{jt}C_{jt}}{\mathrm{P}_tC_t} + \sum_{i=1}^{N} \omega_{ijt} \frac{P_{jt}}{\mathrm{P}_tC_t} \frac{P_{it}^M M_{it}}{P_{jt}}.$$

Since $\gamma_{it} = P_{jt}C_{jt}/(\mathrm{P_t}C_t)$ and $(1-\alpha_i) = P^M_{it}M_{it}/(P_{it}Q_{it})$, we finally have that:

$$\lambda_{jt} = \gamma_{it} + \sum_{i=1}^{N}(1-\alpha_i)\omega_{ijt}\lambda_{it}.$$

B.5 Size-Centrality Relationship in Two Simple Economies

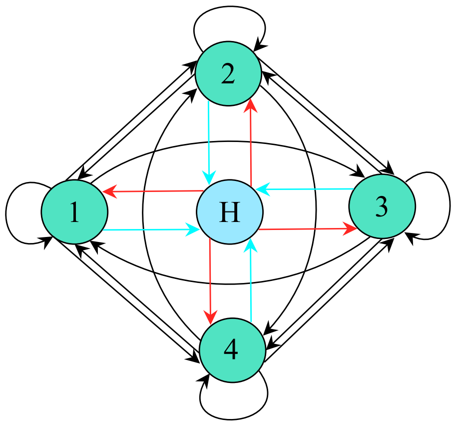
Consider two simple economies consisting of four industries and a representative consumer. In both examples, we assume that (i) the labour share in gross output equals 40% for all industries (i.e. $\alpha_i = \alpha = 0.4$), (ii) there are slightly decreasing returns to scale ($\eta_i = 0.95$), implying that preference shocks affect relative prices, (iii) intermediate inputs are gross complements ($\varepsilon_M = 0.4$), and (iv) the Frisch elasticity of labour supply equals 2 (i.e. $\phi = 1.5$). The two crucial parameters for the purposes of this exercise are ε_M and η_i: $\varepsilon_M < 1$ implies that shocks to industry j that lower (raise) its price will lower (raise) its centrality as measured by its outdegree, and $\eta_i < 1$ implies that relative prices (and thus also outdegrees) need not be invariant to preference shocks.[56]

Example 1: Symmetric Economy

Consider a $N = 4$ economy consisting of identical industries and a representative household, i.e. it is perfectly symmetric (Figure B.1). In particular, assume that the steady-state intermediate input shares $\mu_{ij} = 1/4$ for all (i, j) pairs, and that the steady-state consumption shares $\xi_i = 1/4$ for all i. We further assume that all shocks follow an AR(1) process, and consider separately a 20% increase in Z_{1t} at time $t = 1$, and a 20% increase in D_{1t} at time $t = 1$. The effect of these innovations will die out gradually as we set the AR(1) coefficient to 0.9.

[56]If $\eta_i = 1$, preference shocks will have no effect on relative prices. If $\varepsilon_M = 1$, industries' centralities will be independent of shocks.

Figure B.1. Graphical representation of the economy in Example 1



Notes: Input-output linkages are denoted with black arrows. The supply of labour from the household to producers is denoted with red arrows. The supply of consumption goods to the household is denoted with cyan arrows.

Figure B.2 shows the responses to the positive technology shock to industry 1. This shock will make industry 1 relatively more productive than the other industries, driving its real output up and its price down. Its outdegree and Domar weight fall, and the opposite happens to the other industries, whose responses are identical given the symmetric nature of this economy. Intuitively, since P_{1t} falls by more than the other prices upon impact, the outdegree of industry 1 falls. Although the real output of all industries increases, nominal output of industry 1 increases by less due to the offsetting effect of its price falling by more, and hence its Domar weight falls upon impact. In other words, idiosyncratic technology shocks imply a negative (positive) relationship between output (Domar weights) and centrality for the industry in which the shock originates. For the other industries, the implied size-centrality relationship is positive.

Figure B.2. Responses to a positive technology shock in industry 1 (Example 1)

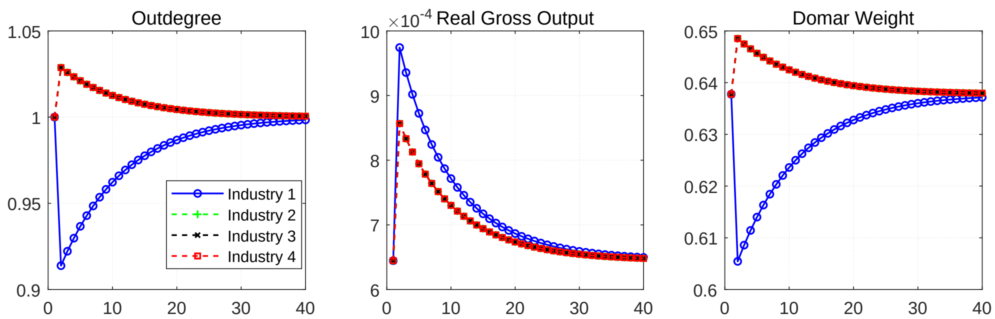
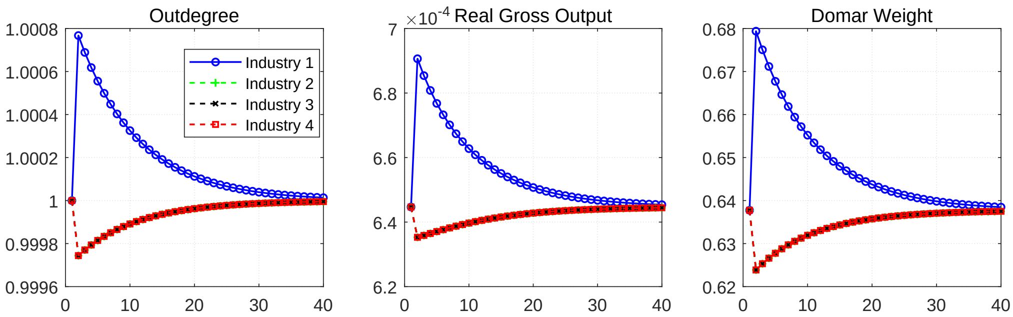


Figure B.3 shows the responses to the positive demand shock in industry 1. The exogenous increase in the household's 'taste' for the output of industry 1 will drive its real output and price up, so industry 1's outdegree, real output, and Domar weight all increase. The opposite happens to the other industries. Intuitively, since P_{1t} increases by more than the other prices, industry 1's outdegree increases. Therefore, idiosyncratic preference shocks imply a positive size-centrality relationship for all industries. Note, however, that although the demand shock in industry 1 increases its outdegree, the effect is very small. Intuitively, demand shocks leave relative prices unchanged in this economy *as long as* there are constant returns to scale.[57] Since we assume slightly decreasing returns to scale ($\eta = 0.95$), demand shocks do affect prices, but only slightly.

Figure B.3. Responses to a positive demand shock in industry 1 (Example 1)

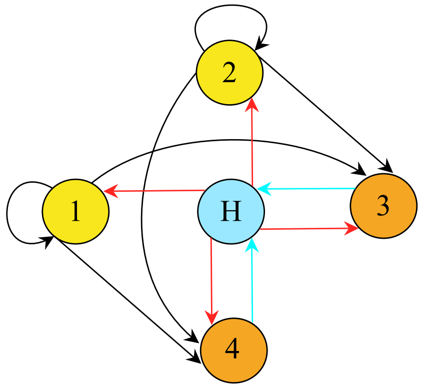


Example 2: Separate Production of Intermediate and Final Goods

Consider now a different $N = 4$ economy consisting of industries 1 and 2, which produce intermediate goods *only* without sourcing from each other, and industries 3 and 4, which produce final goods *only* and source their intermediate inputs from industries 1 and 2 (Figure B.4). Suppose further that industry 2 is relatively more important as an input supplier than industry 1. We will parametrise this economy by setting the intermediate input shares $\mu_{11} = \mu_{22} = 1$, $\mu_{31} = \mu_{41} = 0.3$, $\mu_{32} = \mu_{42} = 0.7$, and $\mu_{ij} = 0$ for all other (i, j) pairs. Assume further that industry 3 is relatively less important as a final good producer than industry 4, i.e. $\xi_3 = 0.3$. We first consider a 20% increase in Z_{1t} at time $t = 1$, and then separately a 20% increase in D_{3t} at time $t = 1$.

[57] If $\eta_i = 1$, we can solve for prices in equation (14) (taking W_t as the numeraire) independently of the demand shocks.

Figure B.4. Graphical representation of the economy in Example 2



Notes: Input-output linkages are denoted with black arrows. The supply of labour from the household to producers is denoted with red arrows. The supply of consumption goods to the household is denoted with cyan arrows.

Figure B.5 shows the responses to a positive technology shock to industry 1. This shock makes one of the two input-supplying industries relatively more productive than before, reducing its marginal cost and allowing it to set a lower price. Since industry 1's price falls by more than that of industry 2, industry 1's outdegree falls and that of industry 2 (equal to $4 - D_{1t}^{out}$) increases. Final-good-producing industries 3 and 4 have, by construction, zero outdegrees. Real output increases across all industries. The Domar weight of industry 1 falls and that of industry 2 increases. The Domar weights of the final good producers are unchanged since they only depend on consumer preferences:

$$\frac{P_{it}Q_{it}}{\mathrm{P}_t C_t} = \frac{P_{it}C_{it}}{\mathrm{P}_t C_t} = \gamma_{it}, \qquad i = 3, 4. \tag{79}$$

which is a consequence of the Cobb-Douglas nature of consumer preferences. Therefore, in this economy which features a completely separate production of intermediate and final goods, the implied size-centrality relationship due to technology shocks to intermediate producers is the same as in the symmetric economy in example 1.[58]

[58]The final-good producers in this economy have, by construction, zero centrality. Since technology shocks to them do not affect the relative price of the intermediate-good producers, they imply no relationship between size and centrality for any industry.

Figure B.5. Responses to a positive technology shock in industry 1 (Example 2)

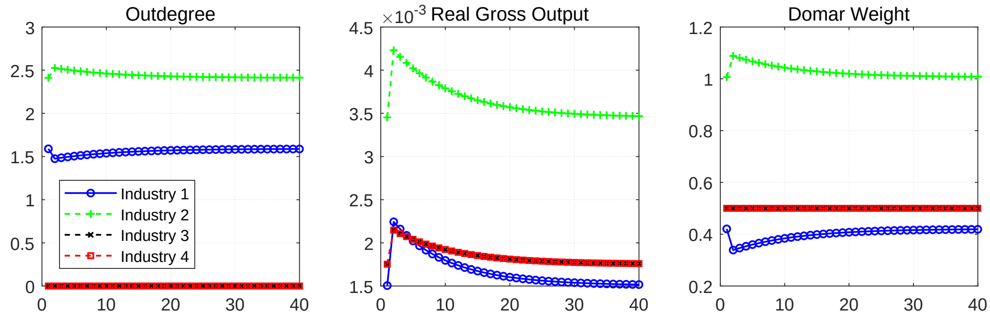
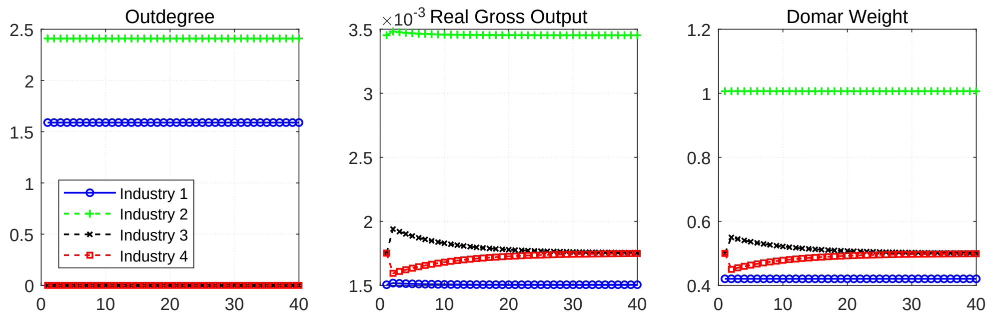


Figure B.6 shows the impulse responses to a positive demand shock in industry 3. This shock increases the consumer's taste for the output of industry 3. Note that the outdegrees of all industries are unchanged. Intuitively, the outdegrees of industries 1 and 2 are a function of P_{1t} and P_{2t} only (and some parameters), and the relative price P_{1t}/P_{2t} is constant as long as there are no technology shocks to industries 1 and 2. The figure also shows that real output of industry 3 increases, whilst that of industry 4 falls; this is expected given the 'shift' in consumer preferences. Real gross output of the input-supplying industries 1 and 2 increases slighly. Finally, whilst the Domar weight of industry 3 increases and that of industry 4 falls, the Domar weights of the input suppliers are unchanged. Note that the Domar weights of industries 1 and 2 are independent of all shocks in the model, and do not change because the relative price P_{1t}/P_{2t} is independent of demand shocks.[59] Therefore, if the production of intermediate and final goods is completely separated, demand shocks imply no relationship between size and centrality (since centrality is independent of them).

Figure B.6. Responses to a positive demand shock in industry 3 (Example 2)



[59]If there are also constant returns to scale, the Domar weights of industries 1 and 2 sum to $(1-\alpha)/\alpha$.

Overall, these two examples based on the simple model from Section 3 suggest that, in general, technology shocks tend to imply a negative relationship between real output and outdegrees, and a positive one between Domar weights and outdegrees.[60] Demand shocks, on the other hand, tend to imply a positive size-centrality relationship regardless of whether one takes output or Domar weight as the measure of size.

B.6 Two Generalisations of the Elasticity of Substitution Across Intermediate Inputs

Generalisation 1: Different ε_M Across Producers

We implicitly assumed above that ε_M is identical across all producers. Allowing ε_M to differ across producers would simply require replacing ε_M by the producer-specific ε_M^i in equation (16).[61] This is because the only elasticities that appear in the cost-minimisation problem of each producer are those characterising its own production process. The effect of changing P_{jt} on P_{it}^M in equation (16) will depend on ε_M^i (and not ε_M^j), but since the weight of P_{jt} in P_{it}^M will typically be less than one, the change in P_{it}^M will tend to be less than proportional, at least in the range of empirically plausible values of ε_M^i. The intuition from above would thus apply: as long as $\varepsilon_M^i < 1$, a shock that causes industry j's own price P_{jt} to increase (decrease) will lead to an increase (decrease) in industry j's outdegree.

Generalisation 2: More vs. Less Substitutable Intermediate Inputs

Another assumption that we made above is that within the bundle of intermediate inputs (M_{it}), all inputs are equally substitutable (or complementary). One might instead argue that, for a given producer, there may exist multiple kinds of intermediate inputs. More specifically, suppose that each producer faces a set of (almost) essential intermediate inputs, $\mathcal{S}$, and a set consisting of the remaining intermediate inputs, $\mathcal{N}$. Intermediate inputs such as electricity, gas, and water could plausibly belong to the former set, and others such as water transport and air transport could belong to the latter set. Producers would then bundle together the two sets of inputs, each consisting of (different) intermediate inputs themselves. Assuming the bundling is consistent with CES aggregation, the bundle (M_{it}) would be characterised by three elasticities of substitution. First, substitutability within the two sets of inputs would be determined by potentially different elasticities $\varepsilon_{\mathcal{S}}$ and $\varepsilon_{\mathcal{N}}$,

[60] As Example 1 shows, technology shocks may imply a positive size-centrality relationship for the industries *other than* the industry in which the shock originated. However, in more realistic settings such as that in Section 4, the majority of variation in industries' size and centrality will tend to be driven by their *own* shocks, hence our emphasis on the implied relationship for the 'shock-originating' industry.

[61] Similarly, we assume throughout that ε_M is time-invariant. Relaxing this assumption would amount to replacing ε_M by $\varepsilon_{M,t}$ in equation (16), with the underlying logic unchanged.

respectively. Second, the extent to which producers can substitute across the two sets of inputs would be determined by $\varepsilon_{\mathcal{SN}}$.

In Appendix B.7, we show that the first-order weighted outdegree of industry j belonging to set $\varkappa = \{\mathcal{S}, \mathcal{N}\}$ is given by:

$$D_{jt}^{out} = \sum_{i=1}^{N} \mu_{ij} \left(\frac{P_{jt}}{P_{it}^{\varkappa}} \right)^{1-\varepsilon_{\varkappa}} \left(\frac{P_{it}^{\varkappa}}{P_{it}^{M}} \right)^{1-\varepsilon_{\mathcal{SN}}}, \tag{80}$$

where $\varepsilon_{\varkappa}$ corresponds to $\varepsilon_{\mathcal{S}}$ ($\varepsilon_{\mathcal{N}}$) if $j \in \mathcal{S}$ ($j \in \mathcal{N}$), and similarly for $P_{it}^{\varkappa}$, which denotes the ideal price index associated with the bundle of (almost) essential inputs ($\varkappa = \mathcal{S}$) or the remaining inputs ($\varkappa = \mathcal{N}$), respectively.

In Appendix B.7, we show that the partial derivative of industry j's outdegree (where j belongs to $\varkappa = \{\mathcal{S}, \mathcal{N}\}$) with respect to its price is given by:

$$\frac{\partial D_{jt}^{out}}{\partial P_{jt}} = \sum_{i=1}^{N} \mu_{ij} \left[1 - \varepsilon_{\varkappa} - (\varepsilon_{\mathcal{SN}} - \varepsilon_{\varkappa}) \frac{\omega_{ijt}}{\omega_{it}^{\varkappa}} - (1 - \varepsilon_{\mathcal{SN}}) \omega_{ijt} \right] \left(\frac{P_{it}^{\varkappa}}{P_{jt}} \right)^{\varepsilon_{\varkappa}} \left(\frac{P_{it}^{M}}{P_{it}^{\varkappa}} \right)^{\varepsilon_{\mathcal{SN}}} \frac{1}{P_{it}^{M}}, \tag{81}$$

where $\omega_{it}^{\varkappa} = (P_{it}^{\varkappa} M_{it}^{\varkappa}) / (P_{it}^{M} M_{it})$, i.e. the equilibrium share of expenses on the set of $\varkappa$ inputs in industry i's total intermediate consumption. In Appendix B.7, we show that if $\varepsilon_{\mathcal{SN}} < 1$—which seems plausible given our interpretation of the sets $\mathcal{S}$ and $\mathcal{N}$—the value of $\varepsilon_{\varkappa}$ that ensures that industry j's outdegree is increasing in its own price is given by:

$$\varepsilon_{\varkappa} < \frac{\omega_{it}^{\varkappa} - \omega_{ijt} \left[(1 - \varepsilon_{\mathcal{SN}}) \omega_{it}^{\varkappa} + \varepsilon_{\mathcal{SN}} \right]}{\omega_{it}^{\varkappa} - \omega_{ijt}}. \tag{82}$$

Note that the term $(1 - \varepsilon_{\mathcal{SN}}) \omega_{it}^{\varkappa} + \varepsilon_{\mathcal{SN}}$ is smaller than 1 if $\varepsilon_{\mathcal{SN}} < 1$, as we just assumed. The right-hand side in equation (82) is thus *larger* than 1. Therefore, if there exists a set of essential inputs alongside a set of remaining inputs, and if the elasticity of substitution between these two sets of inputs is less than unity, then the positive relationship between industries' outdegrees and their prices is conditional on the elasticity of substitution *within* the set they produce in being strictly less than the value on the right-hand side of (82)—which is a *weaker* requirement than that in equation (17), which required that this elasticity is less than 1.[62] In other words, to the extent that the intermediate aggregation process—featuring essential and less essential inputs—is more realistic than that in the baseline model, the condition that ensures that there exists a positive relationship between outdegrees and own prices is relatively *easier* to satisfy.[63]

[62] We clearly recover this condition if $\varepsilon_{\mathcal{SN}} = 1$.

[63] For example, if $\varepsilon_{\mathcal{SN}} = 0.1$, $\omega_{it}^{\varkappa} = 0.3$, and $\omega_{ijt} = 0.05$, then industry j's outdegree is increasing in its price (P_{jt}) as long as $\varepsilon_{\varkappa} < 1.166$.

B.7 More vs. Less Substitutable Intermediate Inputs

For ease of exposition (and without loss of generality for the sake of argument), assume that $\eta_i = 1$, i.e. all industries produce under constant returns to scale. Relative to the baseline model, we now have that the bundle of intermediate inputs consists of two sets of inputs, some (almost) essential and other which are non-essential, and is given by $M_{it} = \left[\delta_i^{\frac{1}{\varepsilon_{\mathcal{SN}}}} (M_{it}^{\mathcal{S}})^{\frac{\varepsilon_{\mathcal{SN}}-1}{\varepsilon_{\mathcal{SN}}}} + (1-\delta_i)^{\frac{1}{\varepsilon_{\mathcal{SN}}}} (M_{it}^{\mathcal{N}})^{\frac{\varepsilon_{\mathcal{SN}}-1}{\varepsilon_{\mathcal{SN}}}}\right]^{\frac{\varepsilon_{\mathcal{SN}}}{\varepsilon_{\mathcal{SN}}-1}}$. In turn, the bundle of essential inputs is given by $M_{it}^{\mathcal{S}} = \left[\sum_{j\in\mathcal{S}} \tilde{\mu}_{ij}^{\frac{1}{\varepsilon_{\mathcal{S}}}} M_{ijt}^{\frac{\varepsilon_{\mathcal{S}}-1}{\varepsilon_{\mathcal{S}}}}\right]^{\frac{\varepsilon_{\mathcal{S}}}{\varepsilon_{\mathcal{S}}-1}}$ and the non-essential bundle is given by $M_{it}^{\mathcal{N}} = \left[\sum_{j\in\mathcal{N}} \tilde{\mu}_{ij}^{\frac{1}{\varepsilon_{\mathcal{N}}}} M_{ijt}^{\frac{\varepsilon_{\mathcal{N}}-1}{\varepsilon_{\mathcal{N}}}}\right]^{\frac{\varepsilon_{\mathcal{N}}}{\varepsilon_{\mathcal{N}}-1}}$ where $\tilde{\mu}_{ij} = \mu_{ij}/\delta_i$ if $j \in \mathcal{S}$ and $\tilde{\mu}_{ij} = \mu_{ij}/(1-\delta_i)$ if $j \in \mathcal{N}$.

To derive the equation for intermediate input shares (and thus for industries' weighted outdegrees), one may follow the steps from Appendix B.3. The main difference is that, now, there is an FOC with respect to M_{ijt} if $j \in \mathcal{S}$, namely:

$$\begin{aligned} P_{jt} - \lambda_{it}\Bigg[Z_{it}L_{it}^{\alpha_i}\left(\frac{(1-\alpha_i)\varepsilon_{\mathcal{SN}}}{\varepsilon_{\mathcal{SN}}-1} M_{it}^{\frac{(1-\alpha_i)\varepsilon_{\mathcal{SN}}-\varepsilon_{\mathcal{SN}}+1}{\varepsilon_{\mathcal{SN}}}} \delta_i^{\frac{1}{\varepsilon_{\mathcal{SN}}}} \frac{\varepsilon_{\mathcal{SN}}-1}{\varepsilon_{\mathcal{SN}}} (M_{it}^{\mathcal{S}})^{\frac{-1}{\varepsilon_{\mathcal{SN}}}}\right) \\ \left(\frac{\varepsilon_{\mathcal{S}}}{\varepsilon_{\mathcal{S}}-1} (M_{it}^{\mathcal{S}})^{\frac{1}{\varepsilon_{\mathcal{S}}}} \tilde{\mu}_{ij}^{\frac{1}{\varepsilon_{\mathcal{S}}}} \frac{\varepsilon_{\mathcal{S}}-1}{\varepsilon_{\mathcal{S}}} M_{ijt}^{\frac{-1}{\varepsilon_{\mathcal{S}}}}\right)\Bigg] = 0. \end{aligned} \tag{83}$$

Now, the natural price index for M_{it} is given by:

$$P_{it}^{M} = \left[\delta_i (P_{it}^{\mathcal{S}})^{1-\varepsilon_{\mathcal{SN}}} + (1-\delta_i)(P_{it}^{\mathcal{N}})^{1-\varepsilon_{\mathcal{SN}}}\right]^{\frac{1}{1-\varepsilon_{\mathcal{SN}}}},$$

and the two subindices as

$$\begin{aligned} P_{it}^{\mathcal{S}} &= \left(\sum_{j\in\mathcal{S}} \tilde{\mu}_{ij} P_{jt}^{1-\varepsilon_{\mathcal{S}}}\right)^{\frac{1}{1-\varepsilon_{\mathcal{S}}}}, \\ P_{it}^{\mathcal{N}} &= \left(\sum_{j\in\mathcal{N}} \tilde{\mu}_{ij} P_{jt}^{1-\varepsilon_{\mathcal{N}}}\right)^{\frac{1}{1-\varepsilon_{\mathcal{N}}}}. \end{aligned}$$

It can then be shown that:

$$\begin{aligned} P_{it}^{\mathcal{S}} &= \lambda_{it}(1-\alpha_i) Q_{it} M_{it}^{\frac{1-\varepsilon_{\mathcal{SN}}}{\varepsilon_{\mathcal{SN}}}} \delta_i^{\frac{1}{\varepsilon_{\mathcal{SN}}}} (M_{it}^{\mathcal{S}})^{\frac{-1}{\varepsilon_{\mathcal{SN}}}}, \\ P_{it}^{\mathcal{N}} &= \lambda_{it}(1-\alpha_i) Q_{it} M_{it}^{\frac{1-\varepsilon_{\mathcal{SN}}}{\varepsilon_{\mathcal{SN}}}} (1-\delta_i)^{\frac{1}{\varepsilon_{\mathcal{SN}}}} (M_{it}^{\mathcal{N}})^{\frac{-1}{\varepsilon_{\mathcal{SN}}}}. \end{aligned}$$

Following analogous steps to those in B.3, it can be easily shown that the intermediate input shares in equilibrium are now given by:

$$\omega_{ijt} = \begin{cases} \mu_{ij} \left(\frac{P_{jt}}{P_{it}^{\mathcal{S}}}\right)^{1-\varepsilon_{\mathcal{S}}} \left(\frac{P_{jt}^{\mathcal{S}}}{P_{it}^{M}}\right)^{1-\varepsilon_{\mathcal{SN}}} & \text{if } j \in \mathcal{S}, \\ \mu_{ij} \left(\frac{P_{jt}}{P_{it}^{\mathcal{N}}}\right)^{1-\varepsilon_{\mathcal{N}}} \left(\frac{P_{jt}^{\mathcal{N}}}{P_{it}^{M}}\right)^{1-\varepsilon_{\mathcal{SN}}} & \text{if } j \in \mathcal{N}. \end{cases}$$

Without loss of generality, let us focus on $j \in \mathcal{S}$. The first-order weighted outdegree of industry j is given by:

$$D_{jt}^{out} \equiv \sum_{i=1}^{N} \omega_{ijt} = \sum_{i=1}^{N} \mu_{ij} \left(\frac{P_{jt}}{P_{it}^{\mathcal{S}}}\right)^{1-\varepsilon_{\mathcal{S}}} \left(\frac{P_{it}^{\mathcal{S}}}{P_{it}^{M}}\right)^{1-\varepsilon_{\mathcal{SN}}}.$$

Note that the partial derivatives of the price indices corresponding to essential and non-essential inputs with respect to P_{jt} are given by:

$$\begin{aligned} \frac{\partial P_{it}^{\mathcal{S}}}{\partial P_{jt}} &= \frac{1}{1-\varepsilon_{\mathcal{S}}} \left[\sum_{j \in \mathcal{S}} \tilde{\mu}_{ij} P_{jt}^{1-\varepsilon_{\mathcal{S}}}\right]^{\frac{1-1+\varepsilon_{\mathcal{S}}}{1-\varepsilon_{\mathcal{S}}}} \tilde{\mu}_{ij}(1-\varepsilon_{\mathcal{S}}) P_{jt}^{-\varepsilon_{\mathcal{S}}} \\ &= \tilde{\mu}_{ij} \left(\frac{P_{jt}}{P_{it}^{\mathcal{S}}}\right)^{-\varepsilon_{\mathcal{S}}}, \end{aligned}$$

and

$$\begin{aligned} \frac{\partial P_{jt}^{M}}{\partial P_{jt}} &= \frac{1}{1-\varepsilon_{\mathcal{SN}}} (P_{it}^{M})^{\varepsilon_{\mathcal{SN}}} \delta_i (1-\varepsilon_{\mathcal{SN}}) (P_{it}^{\mathcal{S}})^{-\varepsilon_{\mathcal{SN}}} \frac{\partial P_{it}^{\mathcal{S}}}{\partial P_{jt}} \\ &= \delta_i \left(\frac{P_{it}^{\mathcal{S}}}{P_{it}^{M}}\right)^{-\varepsilon_{\mathcal{SN}}} \tilde{\mu}_{ij} \left(\frac{P_{jt}}{P_{it}^{\mathcal{S}}}\right)^{-\varepsilon_{\mathcal{S}}}. \end{aligned}$$

Therefore, we have that the partial derivative of industry j's outdegree with respect to its own price is given by:

$$
\begin{aligned}
\frac{\partial D_{jt}^{out}}{\partial P_{jt}} = \sum_{i=1}^{N} \Bigg[& \mu_{ij}(1-\varepsilon_{\mathcal{S}}) P_{jt}^{-\varepsilon_{\mathcal{S}}} (P_{it}^{\mathcal{S}})^{\varepsilon_{\mathcal{S}}-\varepsilon_{\mathcal{SN}}} (P_{it}^{M})^{\varepsilon_{\mathcal{SN}}-1} \\
& + \mu_{ij} P_{jt}^{1-\varepsilon_{\mathcal{S}}} (\varepsilon_{\mathcal{S}} - \varepsilon_{\mathcal{SN}}) (P_{it}^{\mathcal{S}})^{\varepsilon_{\mathcal{S}}-\varepsilon_{\mathcal{SN}}-1} \frac{\partial P_{it}^{\mathcal{S}}}{\partial P_{jt}} (P_{it}^{M})^{\varepsilon_{\mathcal{SN}}-1} \\
& + \mu_{ij} P_{jt}^{1-\varepsilon_{\mathcal{S}}} (P_{it}^{\mathcal{S}})^{\varepsilon_{\mathcal{S}}-\varepsilon_{\mathcal{SN}}} (\varepsilon_{\mathcal{SN}} - 1) (P_{it}^{M})^{\varepsilon_{\mathcal{SN}}-2} \frac{\partial P_{jt}^{M}}{\partial P_{jt}} \Bigg] \\
= \sum_{i=1}^{N} \Bigg[& \mu_{ij}(1-\varepsilon_{\mathcal{S}}) P_{jt}^{-\varepsilon_{\mathcal{S}}} (P_{it}^{\mathcal{S}})^{\varepsilon_{\mathcal{S}}-\varepsilon_{\mathcal{SN}}} (P_{it}^{M})^{\varepsilon_{\mathcal{SN}}-1} \\
& + \mu_{ij} P_{jt}^{1-\varepsilon_{\mathcal{S}}} (\varepsilon_{\mathcal{S}} - \varepsilon_{\mathcal{SN}}) (P_{it}^{\mathcal{S}})^{\varepsilon_{\mathcal{S}}-\varepsilon_{\mathcal{SN}}-1} \tilde{\mu}_{ij} \left(\frac{P_{jt}}{P_{it}^{\mathcal{S}}} \right)^{-\varepsilon_{\mathcal{S}}} (P_{it}^{M})^{\varepsilon_{\mathcal{SN}}-1} \\
& + \mu_{ij} P_{jt}^{1-\varepsilon_{\mathcal{S}}} (P_{it}^{\mathcal{S}})^{\varepsilon_{\mathcal{S}}-\varepsilon_{\mathcal{SN}}} (\varepsilon_{\mathcal{SN}} - 1) (P_{it}^{M})^{\varepsilon_{\mathcal{SN}}-2} \delta_i \left(\frac{P_{it}^{\mathcal{S}}}{P_{it}^{M}} \right)^{-\varepsilon_{\mathcal{S}}} \tilde{\mu}_{ij} \left(\frac{P_{jt}}{P_{it}^{\mathcal{S}}} \right)^{-\varepsilon_{\mathcal{S}}} \Bigg].
\end{aligned}
$$

Taking out the first term inside the square brackets, we can rewrite this as:

$$
\begin{aligned}
\frac{\partial D_{jt}^{out}}{\partial P_{jt}} = \sum_{i=1}^{N} \Bigg[& \mu_{ij}(1-\varepsilon_{\mathcal{S}}) P_{jt}^{-\varepsilon_{\mathcal{S}}} (P_{it}^{\mathcal{S}})^{\varepsilon_{\mathcal{S}}-\varepsilon_{\mathcal{SN}}} (P_{it}^{M})^{\varepsilon_{\mathcal{SN}}-1} \\
& \Bigg(1 + \frac{\varepsilon_{\mathcal{S}} - \varepsilon_{\mathcal{SN}}}{1-\varepsilon_{\mathcal{S}}} P_{jt} (P_{it}^{\mathcal{S}})^{-1} \tilde{\mu}_{ij} \left(\frac{P_{jt}}{P_{it}^{\mathcal{S}}} \right)^{-\varepsilon_{\mathcal{S}}} \\
& + \frac{\varepsilon_{\mathcal{SN}} - 1}{1-\varepsilon_{\mathcal{S}}} P_{jt} (P_{it}^{M})^{-1} \delta_i \left(\frac{P_{it}^{\mathcal{S}}}{P_{it}^{M}} \right)^{-\varepsilon_{\mathcal{SN}}} \tilde{\mu}_{ij} \left(\frac{P_{jt}}{P_{it}^{\mathcal{S}}} \right)^{-\varepsilon_{\mathcal{S}}} \Bigg) \Bigg].
\end{aligned}
$$

Assuming $\varepsilon_{\mathcal{S}} < 1$, i.e. the essential inputs are relative complements, the sign of $\partial D_{jt}^{out}/\partial P_{jt}$ is thus determined by the sign of the term multiplying it. Define $\omega_{it}^{\mathcal{S}}$ as the share of expenditure on an essential input i in industry i's total expenditure on intermediate inputs, i.e.:

$$
\begin{aligned}
\omega_{it}^{\mathcal{S}} &= \frac{P_{it}^{\mathcal{S}} M_{it}^{\mathcal{S}}}{P_{it}^{M} M_{it}} \\
&= \frac{P_{it}^{\mathcal{S}} \delta_i M_{it} (P_{it}^{\mathcal{S}})^{-\varepsilon_{\mathcal{SN}}} (P_{it}^{M})^{\varepsilon_{\mathcal{SN}}}}{P_{it}^{M} M_{it}} \\
&= \delta_i \left(\frac{P_{it}^{\mathcal{S}}}{P_{it}^{M}} \right)^{1-\varepsilon_{\mathcal{SN}}}.
\end{aligned}
$$

Note that:

$$\begin{aligned}
P_{jt}(P_{it}^{\mathcal{S}})^{-1}\tilde{\mu}_{ij}\left(\frac{P_{jt}}{P_{it}^{\mathcal{S}}}\right)^{-\varepsilon_{\mathcal{S}}} &= \frac{P_{jt}^{1-\varepsilon_{\mathcal{S}}}}{(P_{it}^{\mathcal{S}})^{1-\varepsilon_{\mathcal{S}}}}\tilde{\mu}_{ij} \\
&= \omega_{ijt}\delta_i^{-1}\left(\frac{P_{it}^{\mathcal{S}}}{P_{it}^{M}}\right)^{\varepsilon_{\mathcal{SN}}-1} \\
&= \frac{\omega_{ijt}}{\omega_{it}^{\mathcal{S}}}.
\end{aligned}$$

Note also that:

$$\begin{aligned}
P_{jt}(P_{it}^{M})^{-1}\delta_i\left(\frac{P_{it}^{\mathcal{S}}}{P_{it}^{M}}\right)^{-\varepsilon_{\mathcal{SN}}}\tilde{\mu}_{ij}\left(\frac{P_{jt}}{P_{it}^{\mathcal{S}}}\right)^{-\varepsilon_{\mathcal{S}}} &= \frac{\omega_{ijt}}{\omega_{it}^{\mathcal{S}}}(P_{it}^{\mathcal{S}})(P_{it}^{M})^{-1}\delta_i\left(\frac{P_{it}^{\mathcal{S}}}{P_{it}^{M}}\right)^{-\varepsilon_{\mathcal{SN}}} \\
&= \frac{\omega_{ijt}}{\omega_{it}^{\mathcal{S}}}\omega_{it}^{\mathcal{S}} = \omega_{ijt}.
\end{aligned}$$

Therefore, we have that:

$$\begin{aligned}
\frac{\partial D_{jt}^{out}}{\partial P_{jt}} &= \sum_{i=1}^{N}\left[\mu_{ij}(1-\varepsilon_{\mathcal{S}})P_{jt}^{-\varepsilon_{\mathcal{S}}}(P_{it}^{\mathcal{S}})^{\varepsilon_{\mathcal{S}}-\varepsilon_{\mathcal{SN}}}(P_{it}^{M})^{\varepsilon_{\mathcal{SN}}-1}\left(1+\frac{\varepsilon_{\mathcal{S}}-\varepsilon_{\mathcal{SN}}}{1-\varepsilon_{\mathcal{S}}}\frac{\omega_{ijt}}{\omega_{it}^{\mathcal{S}}}+\frac{\varepsilon_{\mathcal{SN}}-1}{1-\varepsilon_{\mathcal{S}}}\omega_{ijt}\right)\right] \\
&= \sum_{i=1}^{N}\left[\mu_{ij}P_{jt}^{-\varepsilon_{\mathcal{S}}}(P_{it}^{\mathcal{S}})^{\varepsilon_{\mathcal{S}}-\varepsilon_{\mathcal{SN}}}(P_{it}^{M})^{\varepsilon_{\mathcal{SN}}-1}\left(1-\varepsilon_{\mathcal{S}}-(\varepsilon_{\mathcal{SN}}-\varepsilon_{\mathcal{S}})\frac{\omega_{ijt}}{\omega_{it}^{\mathcal{S}}}-(1-\varepsilon_{\mathcal{SN}})\omega_{ijt}\right)\right].
\end{aligned} \tag{84}$$

If $\varepsilon_{\mathcal{S}} = \varepsilon_{\mathcal{SN}}$, we recover the expression for $\partial D_{jt}^{out}/\partial P_{jt}$ from our baseline model in which the elasticity of substitution across intermediate inputs is the same for all inputs. If $\varepsilon_{\mathcal{S}} \neq \varepsilon_{\mathcal{SN}}$, then a sufficient condition for industry j's first-order weighted outdegree to be increasing in its price if:

$$1-\varepsilon_{\mathcal{S}}-(\varepsilon_{\mathcal{SN}}-\varepsilon_{\mathcal{S}})\frac{\omega_{ijt}}{\omega_{it}^{\mathcal{S}}}-(1-\varepsilon_{\mathcal{SN}})\omega_{ijt} > 0.$$

We can rewrite this inequality as:

$$(1-\varepsilon_{\mathcal{S}})\omega_{it}^{\mathcal{S}}-(\varepsilon_{\mathcal{SN}}-\varepsilon_{\mathcal{S}})\omega_{ijt}-(1-\varepsilon_{\mathcal{SN}})\omega_{ijt}\omega_{it}^{\mathcal{S}} > 0.$$

Collecting the terms involving $\varepsilon_{\mathcal{S}}$ on the left-hand side, we have that:

$$\left(\omega_{ijt}-\omega_{it}^{\mathcal{S}}\right)\varepsilon_{\mathcal{S}} > (1-\varepsilon_{\mathcal{SN}})\omega_{ijt}\omega_{it}^{\mathcal{S}}-\omega_{it}^{\mathcal{S}}+\varepsilon_{\mathcal{SN}}\omega_{ijt}$$

As long as $\omega_{it}^{\mathcal{S}} > \omega_{ijt}$, i.e. that the share of expenditure on essential inputs in total intermediate consumption exceeds the share of an individual good j in total intermediate consumption, we have that industry j's outdegree is increasing in its price if:

$$\varepsilon_{\mathcal{S}} < \frac{\omega_{it}^{\mathcal{S}} - \omega_{ijt}\left[(1-\varepsilon_{\mathcal{SN}})\omega_{it}^{\mathcal{S}} + \varepsilon_{\mathcal{SN}}\right]}{\omega_{it}^{\mathcal{S}} - \omega_{ijt}}. \tag{85}$$

Consider the following three (exhaustive) possibilities. First, the right-hand side of equation (85) equals one if $\varepsilon_{\mathcal{SN}} = 1$. Second, it will be greater than one if $(1-\varepsilon_{\mathcal{SN}})\omega_{it}^{\mathcal{S}} + \varepsilon_{\mathcal{SN}} < 1$. Note that this condition can be equivalently written as $\varepsilon_{\mathcal{SN}}\left(1-\omega_{it}^{\mathcal{S}}\right) < 1-\omega_{it}^{\mathcal{S}}$, which simplifies to the requirement that $\varepsilon_{\mathcal{SN}} < 1$. Third, the right-hand side of equation (85) will be smaller than one if $(1-\varepsilon_{\mathcal{SN}})\omega_{it}^{\mathcal{S}} + \varepsilon_{\mathcal{SN}} > 1$. This condition similarly simplifies to a requirement that $\varepsilon_{\mathcal{SN}} > 1$.

Recall that we assumed earlier that $\varepsilon_{\mathcal{S}} < 1$. Since one may reasonably expect that $\varepsilon_{\mathcal{SN}}$ will also be smaller than one in practice—implying that the sets of essential and non-essential inputs are relative complements—we have that the sufficient condition to ensure that industry j's outdegree is increasing in its price is given by:

$$\varepsilon_{\mathcal{S}} < \frac{\omega_{ijt}\left[(1-\varepsilon_{\mathcal{SN}})\omega_{it}^{\mathcal{S}} + \varepsilon_{\mathcal{SN}}\right] - \omega_{it}^{\mathcal{S}}}{\omega_{ijt} - \omega_{it}^{\mathcal{S}}}, \tag{86}$$

where the right-hand side of inequality (85) is larger than one. In other words, if the 'between-elasticity' ($\varepsilon_{\mathcal{SN}}$) is less than 1, then the within-elasticity ($\varepsilon_{\mathcal{S}}$) may exceed unity but must not be larger than the right-hand-side expression in inequality (86) in order to ensure that industries' first-order weighted outdegrees increase with own prices.

Recall that in deriving the condition given by (85), we focused on the case $j \in \mathcal{S}$. This was without loss of generality and an analogous condition applies for $j \in \mathcal{N}$.

C. Derivations Related to Section 4

Throughout this section, matrices will be denoted in bold. We allow for labour-augmenting technology shocks in these derivations, but we assume them away in Section 4 and instead focus on TFP (and demand) shocks.

C.1 Solving for the Non-Stochastic Steady State

Our model economy is identical to that in Atalay (2017) *except* that we allow for there to (potentially) be non-constant returns to scale. In other words, we allow the values of η_i to be potentially different from one. As a result, the steady state of the economy in Section 4 will potentially be different from that in Atalay (2017) even if all the other parameters had identical values. In solving for the steady state, we will make extensive references to Appendix F in Atalay (2017).

First, we want to show how the cost-minimising solutions for intermediate inputs and the capital-labour bundle change under non-constant returns to scale. The price index of a bundle of intermediate goods in industry J is given by $P_{tJ}^{in} = \left[\sum_{I=1}^{N} \Gamma_{IJ}^{M} P_{tI}^{1-\varepsilon_M}\right]^{\frac{1}{1-\varepsilon_M}}$. Using the first-order condition with respect to $M_{I\to J}$, we have that with a potentially non-unitary η_J (assuming that the steady-state values of technology are $A_J = 1$ for all J):

$$\frac{P_I^{1-\varepsilon_M}}{P_J^{1-\varepsilon_M}} = \left(\frac{\mu_J}{M_J}\right)^{\frac{1-\varepsilon_M}{\varepsilon_Q}} \left(\frac{M_J \Gamma_{IJ}^M}{M_{I\to J}}\right)^{\frac{1-\varepsilon_M}{\varepsilon_M}} \eta_J^{1-\varepsilon_M} Q_J^{(1-\varepsilon_M)\frac{\varepsilon_Q(\eta_J-1)+1}{\eta_J \varepsilon_Q}}.$$

We can rearrange this equation so as to have P_J^{in} on one side, obtaining:

$$P_J^{in} = P_J \left(\frac{\mu_J}{M_J}\right)^{\frac{1}{\varepsilon_Q}} M_J^{\frac{1}{\varepsilon_M}} \eta_J Q_J^{\frac{\varepsilon_Q(\eta_J-1)+1}{\eta_J \varepsilon_Q}} \left[\sum_{I=1}^{N} \Gamma_{IJ}^M \left(\frac{\Gamma_{IJ}^M}{M_{I\to J}}\right)^{\frac{1-\varepsilon_M}{\varepsilon_M}}\right]^{\frac{1}{1-\varepsilon_M}}.$$

Note that, by definition, $M_{tJ} = \left[\sum_{I=1}^{N} (\Gamma_{IJ}^M)^{\frac{1}{\varepsilon_M}} (M_{t,J\to I})^{\frac{\varepsilon_M-1}{\varepsilon_M}}\right]^{\frac{\varepsilon_M}{\varepsilon_M-1}}$. The above equation can thus be simplified to:

$$\left(\frac{P_J^{in}}{P_J}\right)^{1-\varepsilon_Q} = \left(\frac{\mu_J}{M_J}\right)^{\frac{1-\varepsilon_Q}{\varepsilon_Q}} \eta_J^{1-\varepsilon_Q} Q_J^{(1-\varepsilon_Q)\frac{\varepsilon_Q(\eta_J-1)+1}{\eta_J \varepsilon_Q}}. \tag{87}$$

If we have constant returns to scale (i.e. $\eta_J = 1$), then the above equation simplifies to:

$$\left(\frac{P_J^{in}}{P_J}\right)^{1-\varepsilon_Q} = \left(\frac{\mu_J}{M_J}\right)^{\frac{1-\varepsilon_Q}{\varepsilon_Q}} Q_J^{\frac{1-\varepsilon_Q}{\varepsilon_Q}}$$

which corresponds to equation (35) in Appendix F of Atalay (2017). But with a non-unitary η_J, we have that:

$$\left(\frac{P_J^{in}}{P_J}\right)^{1-\varepsilon_Q} \mu_J \eta_J^{\varepsilon_Q - 1} Q_J^{(\varepsilon_Q(\eta_J-1)+1)\frac{(\varepsilon_Q-1)}{\eta_J\varepsilon_Q}} = \mu_J^{\frac{1}{\varepsilon_Q}} M_J^{\frac{\varepsilon_Q-1}{\varepsilon_Q}}. \tag{88}$$

The cost-minimising choice for the capital-labour bundle will also depend on η_J and is given by:

$$(1-\mu_J)^{\frac{1}{\varepsilon_Q}} \left(\left(\frac{K_J}{\alpha_J}\right)^{\alpha_J} \left(\frac{L_J}{1-\alpha_J}\right)^{1-\alpha_j}\right)^{\frac{\varepsilon_Q-1}{\varepsilon_Q}} = \eta_J^{\varepsilon_Q-1}(1-\mu_J) Q_J^{(\varepsilon_Q(\eta_J-1)+1)\frac{(\varepsilon_Q-1)}{\eta_J\varepsilon_Q}} \times \left(\frac{\left(\frac{1-\beta(1-\delta_K)}{\beta}\right)^{\alpha_J} \left[\sum \Gamma_{IJ}^X (P_I)^{1-\varepsilon_X}\right]^{\frac{\alpha_J}{1-\varepsilon_X}}}{P_J}\right)^{1-\varepsilon_Q}. \tag{89}$$

Next, we substitute the cost-minimising solutions for the capital-labour bundle and intermediate inputs—given by equations (88) and (89)—into the production function, which at steady state is given by:

$$Q_J = \left[(1-\mu_J)^{\frac{1}{\varepsilon_Q}} \left(\left(\frac{K_J}{\alpha_J}\right)^{\alpha_J} \left(\frac{L_J}{1-\alpha_J}\right)^{1-\alpha_j}\right)^{\frac{\varepsilon_Q-1}{\varepsilon_Q}} + \mu_J^{\frac{1}{\varepsilon_Q}} (M_J)^{\frac{\varepsilon_Q-1}{\varepsilon_Q}}\right]^{\eta_J\frac{\varepsilon_Q}{\varepsilon_Q-1}},$$

to obtain:

$$Q_J = \left[\eta_J^{\varepsilon_Q-1}(1-\mu_J) Q_J^{(\varepsilon_Q(\eta_J-1)+1)\frac{(\varepsilon_Q-1)}{\eta_J\varepsilon_Q}} \left(\frac{\left(\frac{1-\beta(1-\delta_K)}{\beta}\right)^{\alpha_J} \left[\sum \Gamma_{IJ}^X (P_I)^{1-\varepsilon_X}\right]^{\frac{\alpha_J}{1-\varepsilon_X}}}{P_J}\right)^{1-\varepsilon_Q} + \left(\frac{P_J^{in}}{P_J}\right)^{1-\varepsilon_Q} \mu_J \eta_J^{\varepsilon_Q-1} Q_J^{(\varepsilon_Q(\eta_J-1)+1)\frac{(\varepsilon_Q-1)}{\eta_J\varepsilon_Q}}\right]^{\frac{\eta_J\varepsilon_Q}{\varepsilon_Q-1}}.$$

We can see that Q_J will drop out of the above equation under constant returns to scale, but not under non-constant returns to scale. We can simplify to above equation to obtain:

$$Q_J^{\frac{\varepsilon_Q-1}{\eta_J\varepsilon_Q}\left(1-(\varepsilon_Q(\eta_J-1)+1)\right)} = \eta_J^{\varepsilon_Q-1}(1-\mu_J) \left(\frac{\left(\frac{1-\beta(1-\delta_K)}{\beta}\right)^{\alpha_J} \left[\sum \Gamma_{IJ}^X (P_I)^{1-\varepsilon_X}\right]^{\frac{\alpha_J}{1-\varepsilon_X}}}{P_J}\right)^{1-\varepsilon_Q} + \left(\frac{P_J^{in}}{P_J}\right)^{1-\varepsilon_Q} \mu_J \eta_J^{\varepsilon_Q-1}. \tag{90}$$

Equation (90) gives us a set of N equations in prices and quantities, and we need another set of N equations in order to solve for the steady-state prices and quantities. As in the simple model, we make use of the goods-market clearing condition. The market clearing condition for good J is given by:

$$Q_J = \delta_{C_J} C_J + \sum_{I=1}^{N} (M_{J\to I} + X_{J\to I}). \tag{91}$$

We thus need to write the right-hand side of the above equation in terms of prices and quantities. The consumption of good J will still be given by equation (39) in Appendix F of Atalay (2017). The solution for $M_{J\to I}$ will, however, depend on η_J and we make use of the first-order condition with respect to it, which is given by:

$$M_{J\to I} = (\mu_I)^{\frac{\varepsilon_M}{\varepsilon_Q}} (M_I)^{\frac{\varepsilon_Q-\varepsilon_M}{\varepsilon_Q}} \Gamma_{JI}^M \left(\frac{P_I}{P_J}\right)^{\varepsilon_M} \left(\eta_I Q_I^{\frac{\varepsilon_Q(\eta_I-1)+1}{\eta_I \varepsilon_Q}}\right)^{\varepsilon_M}. \tag{92}$$

Using equation (88), we can substitute in for M_I since:

$$M_I = \left(\frac{P_I^{in}}{P_I}\right)^{-\varepsilon_Q} \mu_I \eta_I^{\varepsilon_Q} Q_I^{\frac{\varepsilon_Q(\eta_I-1)+1}{\eta_I}}.$$

We thus have that:

$$M_{J\to I} = \left(\eta_I Q_I^{\frac{\varepsilon_Q(\eta_I-1)+1}{\eta_I \varepsilon_Q}}\right)^{\varepsilon_Q} \mu_I \Gamma_{JI}^M P_J^{-\varepsilon_M} (P_I^{in})^{\varepsilon_M-\varepsilon_Q} P_I^{\varepsilon_Q}.$$

Next, we solve for the investment input purchases by industry I sourced from industry J. The cost-minimising optimality condition for capital, which equates the rental price of a unit of capital to its marginal revenue product, is given by:

$$\begin{aligned}\frac{1-\beta(1-\delta_K)}{\beta}\left[\sum \Gamma_{IJ}^X (P_I)^{1-\varepsilon_X}\right]^{\frac{1}{1-\varepsilon_X}} &= P_J(1-\mu_J)^{\frac{1}{\varepsilon_Q}} \eta_J Q_J^{\frac{\varepsilon_Q(\eta_J-1)+1}{\eta_J \varepsilon_Q}} \\ &\cdot \left(\frac{K_J}{\alpha_J}\right)^{\alpha_J \frac{\varepsilon_Q-1}{\varepsilon_Q}-1} \left(\frac{L_J}{1-\alpha_J}\right)^{(1-\alpha_J)\frac{\varepsilon_Q-1}{\varepsilon_Q}}.\end{aligned}$$

Using the cost-minimising solution for the capital-labour bundle given by equation (89), the above equation can be simplified to yield:

$$\left(\frac{K_J}{\alpha_J}\right) = \left[\frac{1-\beta(1-\delta_K)}{\beta}\left(\sum \Gamma_{IJ}^X (P_I)^{1-\varepsilon_X}\right)^{\frac{1}{1-\varepsilon_X}}\right]^{-1+\alpha_J(1-\varepsilon_Q)} \eta_J^{\varepsilon_Q} Q_J^{\frac{\varepsilon_Q(\eta_J-1)+1}{\eta_J}} (1-\mu_J)(P_J)^{\varepsilon_Q}. \tag{93}$$

Recall that $X_J = \delta_K K_J$. Using equation (93), we thus have that the cost-minimising solution for the investment input purchases of industry I from industry J is given by:

$$\begin{aligned} X_{J\to I} &= X_I \Gamma^X_{JI} \left(\frac{P_J}{P_I^{inv}}\right)^{-\varepsilon_X} \\ &= X_I \Gamma^X_{JI} (P_J)^{-\varepsilon_X} (P_I^{inv})^{\varepsilon_X} \\ &= \Gamma^X_{JI} (P_J)^{-\varepsilon_X} (1-\mu_I) \eta_I^{\varepsilon_Q} Q_I^{\frac{\varepsilon_Q(\eta_I - 1)+1}{\eta_I}} \alpha_I \delta_K \\ &\quad \cdot \left(\frac{1-\beta(1-\delta_K)}{\beta}\right)^{-1+\alpha_I(1-\varepsilon_Q)} \left[\sum \Gamma^X_{JI} (P_J)^{1-\varepsilon_X}\right]^{\frac{\varepsilon_X - 1 + \alpha_I(1-\varepsilon_Q)}{1-\varepsilon_X}} (P_I)^{\varepsilon_Q}. \end{aligned} \tag{94}$$

Finally, we can substitute the solutions for C_J, $M_{J\to I}$, and $X_{J\to I}$ (given by equation (39) in Appendix F of Atalay (2017), equation (88), and equation (94), respectively) into the market clearing condition for each good J (given by equation (91)) to obtain:

$$Q_J - \sum_{I=1}^{N} \tilde{\Gamma}_{JI} \eta_J^{\varepsilon_Q} Q_I^{\frac{\varepsilon_Q(\eta_I-1)+1}{\eta_I}} = \xi_J (\delta_{C_J})^{\varepsilon_D} \left(\frac{1-\beta(1-\delta_{C_J})}{\beta}\right)^{-\varepsilon_D} (P_J)^{-\varepsilon_D} \bar{C}^{1-\varepsilon_D}, \tag{95}$$

which along with the set of N equations given by equation (90) yields a system of $2N$ equations in $2N$ unknowns, Q_J and P_J for all $J \in (1, ..., N)$, which we can solve using linear algebra. More specifically, we solve this system of equations as follows. First, we use equation (90) to express Q_J as a function of prices only. We then substitute for quantities in equation (93) and solve the resulting system of N equations in N unknowns.

In terms of the steady-state shares derived on page 55 in Appendix F of Atalay (2017), introducing potentially non-constant returns to scale implies that the first equation (equation (42) in Atalay's (2017) Appendix F) will change. To see why, start from the FOC w.r.t. L_J and substitute in the cost-minimising solution for the capital-labour bundle to get:

$$1 = P_J^{\varepsilon_Q} \eta_J^{\varepsilon_Q} (1-\mu_J) Q_J^{\frac{\varepsilon_Q(\eta_J-1)+1}{\eta_J}} \left[\left(\frac{1-\beta(1-\delta_K)}{\beta}\right) P_J^{inv}\right]^{\alpha_J(1-\varepsilon_Q)}.$$

The next two equations (with C_J and S_I^C on the left-hand side, respectively) are invariant to the degree of returns to scale. The subsequent equation, with $M_{J\to I}/Q_J$ on the left-hand side, will change to:

$$\frac{M_{J\to I}}{Q_J} = \frac{1}{Q_J} \eta_I^{\varepsilon_Q} Q_I^{\frac{\varepsilon_Q(\eta_I-1)+1}{\eta_I}} \mu_I \Gamma^M_{JI} P_J^{-\varepsilon_M} (P_I^{in})^{\varepsilon_M - \varepsilon_Q} P_I^{\varepsilon_Q}.$$

The equation with $X_{J\to I}/Q_J$ on the left-hand side will similarly change, with Q_I in the equation Atalay (2017) derives replaced by $\eta_I^{\varepsilon_Q} Q_I^{\frac{\varepsilon_Q(\eta_I-1)+1}{\eta_I}}$. Finally, equations (44) and (45) on page 55 of Atalay (2017) will remain unchanged regardless of whether $\eta_I = 1$ for all I.

C.2 Log-Linearisation of the Model Equilibrium

We now log-linearise the model equilibrium conditions. Relaxing the assumption of constant returns to scale will imply the following changes to the log-linear equations derived in Appendix F.2 in Atalay (2017). First, the equation for the relative price between industries I and J will change to:

$$\hat{p}_{tI} - \hat{p}_{tJ} = \frac{\varepsilon_Q - 1}{\varepsilon_Q}\hat{a}_{tJ} + \frac{\varepsilon_Q(\eta_J - 1) + 1}{\eta_J \varepsilon_Q}\hat{q}_{tJ} + \left(\frac{1}{\varepsilon_M} - \frac{1}{\varepsilon_Q}\right)\hat{m}_{tJ} - \frac{1}{\varepsilon_M}\hat{m}_{t,I\to J},$$

i.e. the coefficient multiplying $\hat{q}_{tJ}$ will change. Second, the equation for labour will change analogously, with the coefficient multiplying $\hat{q}_{tJ}$ changing from $\frac{1}{\varepsilon_Q}$ to $\frac{\varepsilon_Q(\eta_J-1)+1}{\eta_J\varepsilon_Q}$:

$$\begin{aligned}\frac{1}{\varepsilon_{LS}}\sum_{J'=1}^{N} S^L_{J'}\hat{l}_{tJ'} = \hat{p}_{tJ} + \frac{\varepsilon_Q - 1}{\varepsilon_Q}\hat{a}_{tJ} + \frac{(\varepsilon_Q - 1)(1-\alpha_J)}{\varepsilon_Q}\hat{b}_{tJ} + \frac{\varepsilon_Q(\eta_J - 1) + 1}{\eta_J\varepsilon_Q}\hat{q}_{tJ} \\ + \alpha_J\frac{\varepsilon_Q - 1}{\varepsilon_Q}\hat{k}_{tJ} + \varepsilon\alpha_J - 1 - \alpha_J\varepsilon_Q\varepsilon_Q\hat{l}_{tJ}.\end{aligned}$$

Third, the log-linearised FOC w.r.t. $K_{t+1,J}$ will change similarly to:

$$\begin{aligned}\frac{1}{1-\beta(1-\delta_K)}\hat{p}^{inv}_{tJ} - \frac{\beta(1-\delta_K)}{1-\beta(1-\delta_K)}\hat{p}^{inv}_{t+1,J} = \hat{p}_{t+1,J} + \frac{\varepsilon_Q(\eta_J - 1) + 1}{\eta_J\varepsilon_Q}\hat{q}_{t+1,J} \\ + \frac{\varepsilon_Q - 1}{\varepsilon_Q}\hat{a}_{t+1,J} + \frac{(\varepsilon_Q - 1)(1-\alpha_J)}{\varepsilon_Q}\hat{b}_{t+1,J} \\ + \frac{(\varepsilon_Q - 1)(1-\alpha_J)}{\varepsilon_Q}\hat{l}_{t+1,J} + \left[-1 + \frac{\alpha_J(\varepsilon_Q - 1)}{\varepsilon_Q}\right]\hat{k}_{t+1,J}.\end{aligned}$$

Finally, the log-linearized form of the production function will change to:

$$\hat{q}_{tJ} = \eta_J\left[\hat{\alpha}_{tJ} + \alpha_J(1 - S_{M_J})\hat{k}_{tJ} + (1-\alpha_J)(1-S_{M_J})\hat{b}_{tJ} + (1-\alpha_J)(1-S_{M_J})\hat{l}_{tJ} + S_{M_J}\hat{m}_{tJ}\right].$$

The remaining equations shown on page 56 in Appendix F.2 of Atalay (2017) will remain unchanged.

Accordingly, the bottom four equations on page 57 of Appendix F.2 of Atalay (2017) will change. Let Σ denote diag(η), where η denotes the $N \times 1$ vector of η_J's. Then, we will have that in the equation with $\hat{m}_t$ on the left-hand side, the coefficient on $\hat{q}_t$ on the right-hand side will change to $\frac{\varepsilon_M}{\varepsilon_Q}\left[\varepsilon_Q\left(\Sigma - \mathbf{I}\right) + \mathbf{I}\right]\Sigma^{-1}T_1$. Similarly, in the equations with $\frac{1}{\varepsilon_{LS}}S^L\hat{l}_t$ and $\hat{p}^{inv}_t$ on the left-hand side, the coefficient on $\hat{q}_t$ on the right-hand side will change to $\frac{1}{\varepsilon_Q}\left[\varepsilon_Q\left(\Sigma - \mathbf{I}\right) + \mathbf{I}\right]\Sigma^{-1}$. Finally, we have that Σ will premultiply all terms on the right-hand side in the last equation (corresponding to the log-linearised production function in matrix form):

$$\hat{q}_t = \Sigma \left[\hat{a}_t + (\mathbf{I} - \alpha)(\mathbf{I} - S_M)\hat{b}_t + \alpha(\mathbf{I} - S_M)\hat{k}_t + (\mathbf{I} - \alpha)(\mathbf{I} - S_M)\hat{l}_t + S_M \hat{\mathbb{M}}_t \right],$$

where $\hat{\mathbb{M}}_t = (\varepsilon_Q - 1)\hat{a}_t + [\varepsilon_Q (\Sigma - \mathbf{I}) + \mathbf{I}] \Sigma^{-1}\hat{q}_t + \varepsilon_Q(\mathbf{I} - S_1^M)\hat{p}_t$. Substituting in for $\hat{\mathbb{M}}_t$, collecting the terms involving $\hat{q}_t$ on the left-hand side, and solving for $\hat{q}_t$ yields:

$$\begin{aligned} \hat{q}_t = {}& \left[\Sigma^{-1} - S_M \left[\varepsilon_Q(\Sigma - I) + I\right] \Sigma^{-1}\right]^{-1} \Bigg[(\mathbf{I} + S_M(\varepsilon_Q - 1))\hat{a}_t + (\mathbf{I} - \alpha)(\mathbf{I} - S_M)\hat{b}_t \\ & + \alpha(\mathbf{I} - S_M)\hat{k}_t + (\mathbf{I} - \alpha)(\mathbf{I} - S_M)\hat{l}_t + S_M\varepsilon_Q(\mathbf{I} - S_1^M)\hat{p}_t \Bigg]. \end{aligned} \tag{96}$$

We can then plug the above equation for $\hat{q}_t$ into the remaining log-linear equations shown in *Step 3* on page 59 in Atalay's (2017) Appendix F. First, we have that in the equation involving $\hat{c}_t$ and $\hat{c}_{t+1}$ (inter alia):

$$\begin{aligned} 0 = {}& \delta_C^{-1}\tilde{S}_C^Q\hat{c}_{t+1} + (\mathbf{I} - \delta_C^{-1})\tilde{S}_C^Q\hat{c}_t + (\varepsilon_Q - 1)\tilde{S}_M^Q T_1\hat{a}_t + \tilde{S}_X^Q T_1\delta_K^{-1}\hat{k}_{t+1} + \tilde{S}_X^Q T_1(1 - \delta_K^{-1})\hat{k}_t \\ & + \left[\varepsilon_Q\tilde{S}_M^Q T_1(\mathbf{I} - S_1^M) + \varepsilon_M\tilde{S}_M^Q[T_1S_1^M - T_2] + \varepsilon_X\tilde{S}_X^Q[T_1S_1^X - T_2]\right]\hat{p}_t \\ & - \left(\mathbf{I} - \tilde{S}_M^Q T_1\right)\left[\Sigma^{-1} - S_M\left[\varepsilon_Q(\Sigma - I) + I\right]\Sigma^{-1}\right]^{-1} \Bigg[(\mathbf{I} + S_M(\varepsilon_Q - 1))\hat{a}_t \\ & + (\mathbf{I} - \alpha)(\mathbf{I} - S_M)\hat{b}_t + \alpha(\mathbf{I} - S_M)\hat{k}_t + (\mathbf{I} - \alpha)(\mathbf{I} - S_M)\hat{l}_t + S_M\varepsilon_Q(\mathbf{I} - S_1^M)\hat{p}_t \Bigg]. \end{aligned}$$

Collecting terms, we have that:

$$\begin{aligned} 0 = {}& \delta_C^{-1}\tilde{S}_C^Q\hat{c}_{t+1} + (\mathbf{I} - \delta_C^{-1})\tilde{S}_C^Q\hat{c}_t \\ & + \left[(\varepsilon_Q - 1)\tilde{S}_M^Q T_1 - \left(\mathbf{I} - \tilde{S}_M^Q T_1\right)\left[\Sigma^{-1} - S_M\left[\varepsilon_Q(\Sigma - I) + I\right]\Sigma^{-1}\right]^{-1}(\mathbf{I} + S_M(\varepsilon_Q - 1))\right]\hat{a}_t \\ & - \left[\left(\mathbf{I} - \tilde{S}_M^Q T_1\right)\left[\Sigma^{-1} - S_M\left[\varepsilon_Q(\Sigma - I) + I\right]\Sigma^{-1}\right]^{-1}(\mathbf{I} - \alpha)(\mathbf{I} - S_M)\right]\hat{b}_t \\ & + \tilde{S}_X^Q T_1\delta_K^{-1}\hat{k}_{t+1} + \left[\tilde{S}_X^Q T_1(1 - \delta_K^{-1}) - \left(\mathbf{I} - \tilde{S}_M^Q T_1\right)\left[\Sigma^{-1} - S_M\left[\varepsilon_Q(\Sigma - I) + I\right]\Sigma^{-1}\right]^{-1}\alpha(\mathbf{I} - S_M)\right]\hat{k}_t \\ & - \left\{\left(\mathbf{I} - \tilde{S}_M^Q T_1\right)\left[\Sigma^{-1} - S_M\left[\varepsilon_Q(\Sigma - I) + I\right]\Sigma^{-1}\right]^{-1}(\mathbf{I} - \alpha)(\mathbf{I} - S_M)\right\}\hat{l}_t \\ & + \Bigg[\varepsilon_Q\tilde{S}_M^Q T_1(\mathbf{I} - S_1^M) + \varepsilon_M\tilde{S}_M^Q[T_1S_1^M - T_2] + \varepsilon_X\tilde{S}_X^Q[T_1S_1^X - T_2] \\ & - \left(\mathbf{I} - \tilde{S}_M^Q T_1\right)\left[\Sigma^{-1} - S_M\left[\varepsilon_Q(\Sigma - I) + I\right]\Sigma^{-1}\right]^{-1} S_M\varepsilon_Q(\mathbf{I} - S_1^M)\Bigg]\hat{p}_t. \end{aligned}$$

We also have the following equation, which is unaffected by the degree of returns to scale:

$$\hat{p}_t = \beta(\mathbf{I} - \delta_C)\hat{p}_{t+1} - \frac{1}{\varepsilon_D}\left(\mathbf{I} - \beta(\mathbf{I} - \delta_C)\right)\left[\mathbf{I} - S_I^C(\varepsilon_D - 1)\right]\hat{c}_{t+1}.$$

We also have the following equation:

$$\begin{aligned} S_1^X \hat{p}_t = &\left[\tilde{\beta}\mathbf{I} + \beta(1-\delta_K)S_1^X\right]\hat{p}_{t+1} + \frac{\tilde{\beta}}{\varepsilon_Q}\left[\varepsilon_Q(\Sigma - I) + I\right]\Sigma^{-1}\hat{q}_{t+1} \\ &+ \tilde{\beta}\frac{\varepsilon_Q - 1}{\varepsilon_Q}\hat{a}_{t+1} + \tilde{\beta}\left(-\mathbf{I} + \alpha\frac{\varepsilon_Q - 1}{\varepsilon_Q}\right)\hat{k}_{t+1} + \tilde{\beta}(\mathbf{I} - \alpha)\frac{\varepsilon_Q - 1}{\varepsilon_Q}(\hat{l}_{t+1} + \hat{b}_{t+1}). \end{aligned}$$

Plugging in for $\hat{q}_t$ from equation (96), we have that:

$$\begin{aligned} S_1^X \hat{p}_t = &\left[\tilde{\beta}\mathbf{I} + \beta(1-\delta_K)S_1^X + \mho S_M \varepsilon_Q(\mathbf{I} - S_1^M)\right]\hat{p}_{t+1} \\ &+ \left[\tilde{\beta}\frac{\varepsilon_Q - 1}{\varepsilon_Q} + \mho(\mathbf{I} + S_M(\varepsilon_Q - 1))\right]\hat{a}_{t+1} \\ &+ \left[\tilde{\beta}\left(-\mathbf{I} + \alpha\frac{\varepsilon_Q - 1}{\varepsilon_Q}\right) + \mho\alpha(\mathbf{I} - S_M)\right]\hat{k}_{t+1} \\ &+ \left[\tilde{\beta}(\mathbf{I} - \alpha)\frac{\varepsilon_Q - 1}{\varepsilon_Q} + \mho(\mathbf{I} - \alpha)(\mathbf{I} - S_M)\right]\hat{b}_{t+1} \\ &+ \left[\tilde{\beta}(\mathbf{I} - \alpha)\frac{\varepsilon_Q - 1}{\varepsilon_Q} + \mho(\mathbf{I} - \alpha)(\mathbf{I} - S_M)\right]\hat{l}_{t+1} \end{aligned}$$

where $\mho \equiv \frac{\tilde{\beta}}{\varepsilon_Q}\left[\varepsilon_Q(\Sigma - I) + I\right]\Sigma^{-1}\left[\Sigma^{-1} - S_M\left[\varepsilon_Q(\Sigma - I) + I\right]\Sigma^{-1}\right]^{-1}$.

Finally, in the equation involving $\hat{l}_t$, we have that:

$$\begin{aligned} \frac{1}{\varepsilon_{LS}}S^L\hat{l}_t = &\hat{p}_t + \frac{\varepsilon_Q - 1}{\varepsilon_Q}\hat{a}_t + \frac{(\varepsilon_Q - 1)(I - \alpha)}{\varepsilon_Q}\hat{b}_t \\ &+ \frac{1}{\varepsilon_Q}\left[\varepsilon_Q(\Sigma - I) + I\right]\Sigma^{-1}\hat{q}_t + \frac{\varepsilon_Q - 1}{\varepsilon_Q}\alpha\hat{k}_t + \frac{\alpha - 1 - \alpha\varepsilon_Q}{\varepsilon_Q}\hat{l}_t, \end{aligned}$$

which we can rewrite by substituting in for $\hat{q}_t$ using equation (96). Letting $\Theta \equiv \left[\Sigma^{-1} - S_M\left[\varepsilon_Q(\Sigma - I) + I\right]\Sigma^{-1}\right]^{-1}$, we have that:

$$\begin{aligned} \hat{q}_t = \Theta\Big[&(\mathbf{I} + S_M(\varepsilon_Q - 1))\hat{a}_t + (\mathbf{I} - \alpha)(\mathbf{I} - S_M)\hat{b}_t + \alpha(\mathbf{I} - S_M)\hat{k}_t \\ &+ (\mathbf{I} - \alpha)(\mathbf{I} - S_M)\hat{l}_t + S_M\varepsilon_Q(\mathbf{I} - S_1^M)\hat{p}_t\Big], \end{aligned} \tag{97}$$

which can be rearranged to yield:

$$\begin{aligned}\left[\frac{1}{\varepsilon_{LS}}S^L - \frac{\alpha - 1 - \alpha\varepsilon_Q}{\varepsilon_Q} - \frac{1}{\varepsilon_Q}\Theta(\mathbf{I}-\alpha)(\mathbf{I}-S_M)\right]\hat{l}_t = &\left[I + \Theta S_M \varepsilon_Q(\mathbf{I}-S_1^M)\right]\hat{p}_t \\ &+ \left[\frac{\varepsilon_Q - 1}{\varepsilon_Q} + \Theta(\mathbf{I} + S_M(\varepsilon_Q - 1))\right]\hat{a}_t \\ &+ \left[\frac{(\varepsilon_Q - 1)(I-\alpha)}{\varepsilon_Q} + \Theta(\mathbf{I}-\alpha)(\mathbf{I}-S_M)\right]\hat{b}_t \\ &+ \left[\frac{\varepsilon_Q - 1}{\varepsilon_Q}\alpha + \Theta\alpha(\mathbf{I}-S_M)\right]\hat{k}_t.\end{aligned}$$

Simplifying the notation, the above equation can be rewritten as:

$$\hat{l}_t = \mathcal{L}_p\hat{p}_t + \mathcal{L}_a\hat{a}_t + \mathcal{L}_b\hat{b}_t + \mathcal{L}_k\hat{k}_t. \tag{98}$$

We can similarly simplify the notation in the remaining three equations:

$$0 = \mathcal{C}_c^1\hat{c}_{t+1} + \mathcal{C}_c\hat{c}_t + \mathcal{C}_a\hat{a}_t + \mathcal{C}_b\hat{b}_t + \mathcal{C}_{k+}\hat{k}_{t+1} + \mathcal{C}_k\hat{k}_t + \mathcal{C}_p\hat{p}_t + \mathcal{C}_l\hat{l}_t, \tag{99}$$

$$\hat{p}_t = \beta(\mathbf{I}-\delta_C)\hat{p}_{t+1} - \frac{1}{\varepsilon_D}(\mathbf{I}-\beta(\mathbf{I}-\delta_C))[\mathbf{I}+S_I^C(\varepsilon_D - 1)]\hat{c}_{t+1}, \tag{100}$$

$$S_1^X\hat{p}_t = \mathcal{P}_p^+\hat{p}_{t+1} + \mathcal{P}_a\hat{a}_{t+1} + \mathcal{P}_k\hat{k}_{t+1} + \mathcal{P}_b\hat{b}_{t+1} + \mathcal{P}_l\hat{l}_{t+1}. \tag{101}$$

Substituting for $\hat{l}_t$ using equation (98) into equations (99)-(101), we have that:

$$\begin{aligned}0 &= \mathcal{C}_c^+\hat{c}_{t+1} + \mathcal{C}_c\hat{c}_t + [\mathcal{C}_a + \mathcal{C}_l\mathcal{L}_a]\hat{a}_t + [\mathcal{C}_b + \mathcal{C}_l\mathcal{L}_b]\hat{b}_t + \mathcal{C}_{k+}\hat{k}_{t+1} + [\mathcal{C}_k + \mathcal{C}_l\mathcal{L}_k]\hat{k}_t + [\mathcal{C}_p + \mathcal{C}_l\mathcal{L}_p]\hat{p}_t, \\ \hat{p}_t &= \beta(\mathbf{I}-\delta_C)\hat{p}_{t+1} - \frac{1}{\varepsilon_D}(\mathbf{I}-\beta(\mathbf{I}-\delta_C))[\mathbf{I}+S_I^C(\varepsilon_D - 1)]\hat{c}_{t+1}, \\ S_1^X\hat{p}_t &= \left[\mathcal{P}_p^+ + \mathcal{P}_l\mathcal{L}_p\right]\hat{p}_{t+1} + [\mathcal{P}_a + \mathcal{P}_l\mathcal{L}_a]\hat{a}_{t+1} + [\mathcal{P}_k + \mathcal{P}_l\mathcal{L}_k]\hat{k}_{t+1} + [\mathcal{P}_b + \mathcal{P}_l\mathcal{L}_b]\hat{b}_{t+1}. \end{aligned}\tag{102}$$

Since we are assuming away durable goods (i.e. $\delta_C = 1$), we have from equation (100) that $\hat{c}_{t+1}$ is given by:

$$\hat{c}_{t+1} = -\varepsilon_D[\mathbf{I}+S_I^C(\varepsilon_D - 1)]^{-1}\hat{p}_t. \tag{103}$$

Plugging in for $\hat{c}_{t+1}$ above, we have that:

$$\begin{aligned}0 = &\mathcal{C}_c\hat{c}_t + [\mathcal{C}_a + \mathcal{C}_l\mathcal{L}_a]\hat{a}_t + [\mathcal{C}_b + \mathcal{C}_l\mathcal{L}_b]\hat{b}_t + \mathcal{C}_{k+}\hat{k}_{t+1} + [\mathcal{C}_k + \mathcal{C}_l\mathcal{L}_k]\hat{k}_t \\ &+ \left[-\varepsilon_D\mathcal{C}_c^1[\mathbf{I}+S_I^C(\varepsilon_D - 1)]^{-1} + \mathcal{C}_p + \mathcal{C}_l\mathcal{L}_p\right]\hat{p}_t.\end{aligned}$$

The log-linearized equation for consumption as a function of prices and demand shocks is:[64]

$$\hat{c}_t = \hat{d}_t - \varepsilon_D \left[I + S_I^C(\varepsilon_D - 1)\right]^{-1} \hat{p}_t. \tag{104}$$

By substituting out $\hat{c}_t$ using the above equation, we obtain:

$$\begin{aligned} 0 = &\ \mathcal{C}_c \hat{d}_t + [\mathcal{C}_a + \mathcal{C}_l \mathcal{L}_a]\, \hat{a}_t + [\mathcal{C}_b + \mathcal{C}_l \mathcal{L}_b]\, \hat{b}_t + \mathcal{C}_{k+} \hat{k}_{t+1} + [\mathcal{C}_k + \mathcal{C}_l \mathcal{L}_k]\, \hat{k}_t \\ &+ \left[-\varepsilon_D \left(\mathcal{C}_c^1 + \mathcal{C}_c\right) [\mathbf{I} + S_I^C(\varepsilon_D - 1)]^{-1} + \mathcal{C}_p + \mathcal{C}_l \mathcal{L}_p\right] \hat{p}_t. \end{aligned} \tag{105}$$

Note that Equations (102) and (105) can be written in matrix form as:

$$\begin{bmatrix} \hat{p}_{t+1} \\ \hat{k}_{t+1} \end{bmatrix} = \begin{bmatrix} \mathbf{\Psi}_{pp} & \mathbf{\Psi}_{pk} \\ \mathbf{\Psi}_{kp} & \mathbf{\Psi}_{kk} \end{bmatrix} \begin{bmatrix} \hat{p}_t \\ \hat{k}_t \end{bmatrix} + \begin{bmatrix} \mathbf{\Phi}_{pa} & \mathbf{\Psi}_{pb} & \mathbf{\Psi}_{pd} \\ \mathbf{\Phi}_{ka} & \mathbf{\Psi}_{kb} & \mathbf{\Psi}_{kd} \end{bmatrix} \begin{bmatrix} \hat{a}_t \\ \hat{b}_t \\ \hat{d}_t \end{bmatrix},$$

where the multiplying matrices (in bold) are defined by solving the equations (102) and (105) for $\hat{p}_{t+1}$ and $\hat{k}_{t+1}$, respectively, in terms of $\hat{k}_t$, $\hat{p}_t$, and the shocks ($\hat{a}_t$, $\hat{b}_t$, and $\hat{d}_t$). Using the Blanchard-Kahn decomposition, one can obtain the following two equations (see Appendix F.4 in Atalay (2017)):

$$\hat{k}_{t+1} = M_{kk} \hat{k}_t + [M_{ka}, M_{kb}, M_{kd}] \begin{bmatrix} \hat{a}_t \\ \hat{b}_t \\ \hat{d}_t \end{bmatrix}, \tag{106}$$

$$\hat{p}_t = \Psi_{21}^{-1} \hat{k}_{t+1} - \Psi_{21}^{-1} \Psi_{22} \hat{k}_t - \Psi_{21}^{-1} \Phi_2^d \begin{bmatrix} \hat{a}_t \\ \hat{b}_t \\ \hat{d}_t \end{bmatrix}. \tag{107}$$

By substituting for $\hat{k}_{t+1}$ in equation (107) using equation (106), we can express $\hat{p}_t$ as a function of the state variable, $\hat{k}_t$, and the shocks, i.e.:

$$\hat{p}_t = \mathbf{P}_k \hat{k}_t + \mathbf{P}_a \hat{a}_t + \mathbf{P}_b \hat{b}_t + \mathbf{P}_d \hat{d}_t. \tag{108}$$

Next, by substituting for $\hat{l}_t$ in equation (97) using equation (98), we obtain $\hat{q}_t$ as a function of $\hat{k}_t$, $\hat{p}_t$, and the shocks. We can then in turn plug in for $\hat{p}_t$ (using equation (108) above) and express $\hat{q}_t$ also solely as a function of the state variable, $\hat{k}_t$, and the shocks:

$$\hat{q}_t = \mathbf{\Phi}_k \hat{k}_t + \mathbf{\Phi}_a \hat{a}_t + \mathbf{\Phi}_b \hat{b}_t + \mathbf{\Phi}_d \hat{d}_t. \tag{109}$$

[64] For a derivation of this equation, see page 72 in Appendix F in Atalay (2017).

where the matrices in bold depend (non-linearly) on model parameters only. Taking differences on both sides of equation (109), assuming away labour-augmenting technology shocks (so $\hat{b}_t = 0$ for all t), and denoting $\Delta\hat{a}_t$ by ω^A_{t+1} and $\Delta\hat{d}_t$ by ω^D_{t+1}, we can derive the equation for $\Delta\hat{q}_t$ in the top row of equation (35):

$$\Delta\hat{q}_t = \mathbf{\Phi}_k \Delta\hat{k}_t + \mathbf{\Phi}_a \omega^A_t + \mathbf{\Phi}_d \omega^D_t. \tag{110}$$

C.3 Deriving the log-linearised equation for value-added

By definition, nominal value-added in industry J in period t is given by:

$$VA_{tJ} = P_{tJ}Q_{tJ} - M_{tJ}P^{in}_{tJ}. \tag{111}$$

Recall that the intermediate input bundle, M_{tJ}, is given by equation (27). Recall also that the first-order condition with respect to $M_{J\to I}$ is given by equation (96), as corresponding to the steady state. Adding time-subscripts in that equation and re-introducing shocks (which are normalised to unity in steady state), we have that:

$$M_{t,I\to J}^{\frac{\varepsilon_M-1}{\varepsilon_M}} = \left[\left(\frac{P_{tJ}}{P_{tI}}\right)(A_{tJ})^{\frac{\varepsilon_Q-1}{\varepsilon_Q}}\left(\frac{\mu_J}{M_{tJ}}\right)^{\frac{1}{\varepsilon_Q}}\left(M_{tJ}\Gamma^M_{IJ}\right)^{\frac{1}{\varepsilon_M}}\eta_J Q_{tJ}^{\frac{\varepsilon_Q(\eta_J-1)+1}{\varepsilon_Q\eta_J}}\right]^{\varepsilon_M-1}.$$

Pre-multiplying both sides by $(\Gamma^M_{IJ})^{\frac{1}{\varepsilon_M}}$, summing over all industries $I \in \{1,\ldots,N\}$, and raising the resulting sums on both sides to the power of $\varepsilon_M/(\varepsilon_M - 1)$, one obtains the following equation:

$$M_{tJ} = P^{\varepsilon_M}_{tJ}(A_{tJ})^{\frac{(\varepsilon_Q-1)\varepsilon_M}{\varepsilon_Q}}\left(\frac{\mu_J}{M_{tJ}}\right)^{\frac{\varepsilon_M}{\varepsilon_Q}} M_{tJ}\left[\eta_J Q_{tJ}^{\frac{\varepsilon_Q(\eta_J-1)+1}{\varepsilon_Q\eta_J}}\right]^{\varepsilon_M}(P^{in}_{tJ})^{-\varepsilon_M},$$

which can be rewritten as:

$$M_{tJ} = \left(\frac{P_{tJ}}{P^{in}_{tJ}}\right)^{\varepsilon_Q}(A_{tJ})^{\varepsilon_Q-1}\mu_J\eta_J^{\varepsilon_Q}Q_{tJ}^{\frac{\varepsilon_Q(\eta_J-1)+1}{\eta_J}}. \tag{112}$$

Using the above equation, we can substitute in for M_{tJ} in equation (111), which gives:

$$VA_{tJ} = P_{tJ}Q_{tJ}\left[1 - \left(\frac{P^{in}_{tJ}}{P_{tJ}}\right)^{1-\varepsilon_Q}(A_{tJ})^{\varepsilon_Q-1}\mu_J\eta_J^{\varepsilon_Q}Q_{tJ}^{\frac{\varepsilon_Q(\eta_J-1)+1-\eta_J}{\eta_J}}\right].$$

Therefore, real value-added of industry J in period t (V_{tJ}) is given by:

$$V_{tJ} \equiv \frac{VA_{tJ}}{P_{tJ}} = Q_{tJ}\left[1 - \left(\frac{P^{in}_{tJ}}{P_{tJ}}\right)^{1-\varepsilon_Q}(A_{tJ})^{\varepsilon_Q-1}\mu_J\eta_J^{\varepsilon_Q}Q_{tJ}^{\frac{\varepsilon_Q(\eta_J-1)+1-\eta_J}{\eta_J}}\right]. \tag{113}$$

In order to log-linearize equation (113), we can rewrite it as:

$$\frac{V_{tJ}}{Q_{tJ}} = 1 - \left(\frac{P_{tJ}^{in}}{P_{tJ}}\right)^{1-\varepsilon_Q} (A_{tJ})^{\varepsilon_Q - 1} \mu_J \eta_J^{\varepsilon_Q} Q_{tJ}^{\frac{\varepsilon_Q(\eta_J - 1) + 1 - \eta_J}{\eta_J}},$$

which can in turn be rewritten as:

$$\frac{V_J}{Q_J} e^{\hat{v}_{tJ} - \hat{q}_{tJ}} = 1 - \left(\frac{P_J^{in}}{P_J}\right)^{1-\varepsilon_Q} e^{(1-\varepsilon_Q)(\hat{p}_{tJ}^{in} - \hat{p}_{tJ})} e^{(\varepsilon_Q - 1)\hat{a}_{tJ}} \mu_J \eta_J^{\varepsilon_Q} Q_J^{\frac{\varepsilon_Q(\eta_J - 1) + 1 - \eta_J}{\eta_J}} e^{\left(\frac{\varepsilon_Q(\eta_J - 1) + 1 - \eta_J}{\eta_J}\right)\hat{q}_{tJ}}.$$

To proceed, we make use of the following steady-state relationship, $V_J/Q_J = 1 - \mu_J$, and the approximation that $e^x \approx 1 + x$ if x is small enough. We thus get:

$$\begin{aligned}(1 - \mu_J)(1 + \hat{v}_{tJ} - \hat{q}_{tJ}) = 1 - &\left(\frac{P_J^{in}}{P_J}\right)^{1-\varepsilon_Q} \left[1 + (1 - \varepsilon_Q)\left(\hat{p}_{tJ}^{in} - \hat{p}_{tJ}\right)\right] \\ &\cdot \mu_J \eta_J^{\varepsilon_Q} Q_J^{\frac{\varepsilon_Q(\eta_J - 1) + 1 - \eta_J}{\eta_J}} (1 + (\varepsilon_Q - 1)\hat{a}_t) \left[1 + \left(\frac{\varepsilon_Q(\eta_J - 1) + 1 - \eta_J}{\eta_J}\right)\hat{q}_{tJ}\right].\end{aligned}$$

Next, using the approximation that $\hat{x}_t \hat{y}_t \approx 0$ for any $\hat{x}_t$ and $\hat{y}_t$ denoting (log) deviations of x_t and y_t from their steady-state values, we obtain:

$$\begin{aligned}(1 - \mu_J)(1 + \hat{v}_{tJ} - \hat{q}_{tJ}) = 1 - &\left(\frac{P_J^{in}}{P_J}\right)^{1-\varepsilon_Q} \mu_J \eta_J^{\varepsilon_Q} Q_J^{\frac{\varepsilon_Q(\eta_J - 1) + 1 - \eta_J}{\eta_J}} \\ &\cdot \left[1 + (1 - \varepsilon_Q)\left(\hat{p}_{tJ}^{in} - \hat{p}_{tJ}\right) + (\varepsilon_Q - 1)\hat{a}_t + \left(\frac{\varepsilon_Q(\eta_J - 1) + 1 - \eta_J}{\eta_J}\right)\hat{q}_{tJ}\right].\end{aligned}$$

Note that in steady state—in which all of the variables with a hat in the above equation equal zero—we have that:

$$1 - \mu_J = 1 - \left(\frac{P_J^{in}}{P_J}\right)^{1-\varepsilon_Q} \mu_J \eta_J^{\varepsilon_Q} Q_J^{\frac{\varepsilon_Q(\eta_J - 1) + 1 - \eta_J}{\eta_J}}. \tag{114}$$

Therefore, cancelling out the steady-state terms, we get:

$$\begin{aligned}(1 - \mu_J)(\hat{v}_{tJ} - \hat{q}_{tJ}) = - &\left(\frac{P_J^{in}}{P_J}\right)^{1-\varepsilon_Q} \mu_J \eta_J^{\varepsilon_Q} Q_J^{\frac{\varepsilon_Q(\eta_J - 1) + 1 - \eta_J}{\eta_J}} \\ &\cdot \left[(1 - \varepsilon_Q)\left(\hat{p}_{tJ}^{in} - \hat{p}_{tJ}\right) + (\varepsilon_Q - 1)\hat{a}_t + \left(\frac{\varepsilon_Q(\eta_J - 1) + 1 - \eta_J}{\eta_J}\right)\hat{q}_{tJ}\right]\end{aligned}$$

Taking value-added on the left-hand side, we finally obtain:

$$\begin{aligned}\hat{v}_{tJ} = \hat{q}_{tJ} - &\left(\frac{P_J^{in}}{P_J}\right)^{1-\varepsilon_Q} \frac{\mu_J}{1 - \mu_J} \eta_J^{\varepsilon_Q} Q_J^{\frac{\varepsilon_Q(\eta_J - 1) + 1 - \eta_J}{\eta_J}} \\ &\cdot \left[(1 - \varepsilon_Q)\left(\hat{p}_{tJ}^{in} - \hat{p}_{tJ}\right) + (\varepsilon_Q - 1)\hat{a}_t + \left(\frac{\varepsilon_Q(\eta_J - 1) + 1 - \eta_J}{\eta_J}\right)\hat{q}_{tJ}\right].\end{aligned} \tag{115}$$

Note that in the presence of constant returns to scale ($\eta_J = 1$), equation (115) becomes:

$$\hat{v}_{tJ} = \hat{q}_{tJ} - \frac{\mu_J}{1-\mu_J}\left[(1-\varepsilon_Q)\left[\hat{p}^{in}_{tJ} - \hat{p}_{tJ}\right] + (\varepsilon_Q - 1)\hat{a}_t\right].$$

In matrix form, we can write equation (115) as:

$$\hat{v}_t = \mathcal{V}_q \hat{q}_t + \mathcal{V}_p \hat{p}_t + \mathcal{V}_a \hat{a}_t, \tag{116}$$

where we use the relationship that between the vector of output prices and the prices of intermediate input bundles in each industry.

Recall from equations (108) and (109) that both $\hat{q}_t$ and $\hat{p}_t$ can be expressed as functions of $\hat{k}_t$ and the shocks only. Therefore, we can substitute for $\hat{q}_t$ and $\hat{p}_t$ in equation (116) to express value-added as a function of $\hat{k}_t$ and the shocks. In other words, one obtains:

$$\hat{v}_t = \mathbf{V}_k \hat{k}_t + \mathbf{V}_a \hat{a}_t + \mathbf{V}_b \hat{b}_t + \mathbf{V}_d \hat{d}_t. \tag{117}$$

As long as $\mathbf{V}_k^{-1}$ exists, we can equivalently write the above equation as:

$$\hat{k}_t = \mathbf{V}_k^{-1}\hat{v}_t - \mathbf{V}_k^{-1}\mathbf{V}_a \hat{a}_t - \mathbf{V}_k^{-1}\mathbf{V}_b \hat{b}_t - \mathbf{V}_k^{-1}\mathbf{V}_d \hat{d}_t. \tag{118}$$

We thus have that one period ahead:

$$\begin{aligned}
\hat{v}_{t+1} &= \mathbf{V}_k \hat{k}_{t+1} + \mathbf{V}_a \hat{a}_{t+1} + \mathbf{V}_b \hat{b}_{t+1} + \mathbf{V}_d \hat{d}_{t+1} \\
&= \mathbf{V}_k \left(M_{kk}\hat{k}_t + M_{ka}\hat{a}_t + M_{kb}\hat{b}_t + M_{kd}\hat{d}_t\right) + \mathbf{V}_a \hat{a}_{t+1} + \mathbf{V}_b \hat{b}_{t+1} + \mathbf{V}_d \hat{d}_{t+1}.
\end{aligned}$$

where we make use of equation (106) in the second line. By plugging in for $\hat{k}_t$ in the above equation using equation (118), we therefore obtain $\hat{v}_{t+1}$ as a function of the shocks at time $t+1$ ($\hat{a}_{t+1}$,$\hat{b}_{t+1}$, and $\hat{d}_{t+1}$), the shocks at time t ($\hat{a}_t$,$\hat{b}_t$, and $\hat{d}_t$), and $\hat{v}_t$. Taking differences on both sides of this resulting equation and assuming away labour-augmenting technology shocks, we obtain the top row of equation (38) in the main text:

$$\Delta\hat{v}_{t+1} = \tilde{\mathbf{V}}_v \Delta\hat{v}_t + [\mathbf{V}_a \;\; \mathbf{V}_d]\begin{bmatrix}\omega^A_{t+1} \\ \omega^D_{t+1}\end{bmatrix} + [\tilde{\mathbf{V}}_a \;\; \tilde{\mathbf{V}}_d]\begin{bmatrix}\omega^A_t \\ \omega^D_t\end{bmatrix}. \tag{119}$$

Starting from equation (108) for the vector of prices, $\hat{p}_t$, and following the analogous steps shown above for the vector of value-added (in equation (117)), we obtain the bottom row of equation (38).

C.4 Deriving the log-linear equation for outdegrees

The first-order weighted outdegree of industry J at time t is defined by:

$$d^o_{Jt} = \sum_{I=1}^{N} w_{J\to I,t}.$$

Log-linearizing the above summation, we have that:

$$d^o_{Jt} = \sum_{I=1}^{N} \frac{\omega_{J\to I}}{d_J}\hat{\omega}_{J\to I,t},$$

where

$$\hat{\omega}_{J\to I,t} = \hat{m}_{J\to I,t} + \hat{p}_{J,t} - \hat{m}_{I,t} - \hat{p}^{in}_{I,t}.$$

We can thus show that

$$\hat{d}^o_t = \mathbf{W}\left[\hat{m}_t - T_1\hat{\mathbb{M}}_t + T_2\hat{p}_t - T_1\hat{p}^{int}_t\right],$$

where $\mathbf{W}_{N\times N^2}$ is a matrix that has, e.g., $[w_{1\to 1}/d^o_1 \;\; w_{1\to 2}/d^o_1 \;\; ... \;\; w_{1\to N}/d^o_1 \;\; 0 \;\; \ldots\ldots \;\; 0]$ in its first row, and similarly in the other rows (steady-state values of input shares and outdegrees). Note that above:

$$\hat{m}_t = \begin{bmatrix} \hat{m}_{11t} \\ \hat{m}_{12t} \\ \cdots \\ \hat{m}_{1Nt} \\ \cdots \\ \hat{m}_{N1t} \\ \hat{m}_{N2t} \\ \cdots \\ \hat{m}_{NNt} \end{bmatrix}_{N^2\times 1}, \quad \hat{\mathbb{M}}_t = \begin{bmatrix} \hat{m}_{1t} \\ \hat{m}_{2t} \\ \cdots \\ \hat{m}_{Nt} \end{bmatrix}_{N\times 1}, \quad \hat{p}_t = \begin{bmatrix} \hat{p}_{1t} \\ \hat{p}_{2t} \\ \cdots \\ \hat{p}_{Nt} \end{bmatrix}_{N\times 1}, \quad \hat{p}^{int}_t = \begin{bmatrix} \hat{p}^{int}_{1t} \\ \hat{p}^{int}_{2t} \\ \cdots \\ \hat{p}^{int}_{Nt} \end{bmatrix}_{N\times 1},$$

where the multiplying matrices are given by $T_1 = \mathbf{1}\otimes\mathbf{I}$ and $T_2 = \mathbf{I}\otimes\mathbf{1}$, where $\otimes$ denotes the Kronecker product.

Under the assumption of constant returns to scale, the log-linear equation for the vector of intermediate input bundles in each industry is given by:[65]

$$\hat{m}_t = \frac{\varepsilon_M}{\varepsilon_Q}(\varepsilon_Q - 1)T_1\hat{a}_t + \frac{\varepsilon_M}{\varepsilon_Q}T_1\hat{q}_t + \left(1 - \frac{\varepsilon_M}{\varepsilon_Q}\right) T_1\hat{\mathbb{M}}_t + \varepsilon_M[T_1 - T_2]\hat{p}_t.$$

Instead, allowing for potentially non-constant returns to scale, we will have that:

$$\hat{m}_t = \frac{\varepsilon_M}{\varepsilon_Q}(\varepsilon_Q - 1)T_1\hat{a}_t + \frac{\varepsilon_M}{\varepsilon_Q}T_1\left[\varepsilon_Q(\Sigma - I) + I\right]\Sigma^{-1}\hat{q}_t + \left(1 - \frac{\varepsilon_M}{\varepsilon_Q}\right) T_1\hat{\mathbb{M}}_t + \varepsilon_M[T_1 - T_2]\hat{p}_t. \tag{120}$$

Recall from above that the $N \times 1$ vector of intermediate input bundles ($\hat{\mathbb{M}}_t$) is given by:

$$\hat{\mathbb{M}}_t = (\varepsilon_Q - 1)\hat{a}_t + \left[\varepsilon_Q(\Sigma - I) + I\right]\Sigma^{-1}\hat{q}_t + \varepsilon_Q(\mathbf{I} - S_1^M)\hat{p}_t.$$

Substituting in for $\hat{\mathbb{M}}_t$ in equation (120) and simplifying, we have that:

$$\hat{m}_t = (\varepsilon_Q - 1)T_1\hat{a}_t + T_1\left[\varepsilon_Q(\Sigma - I) + I\right]\Sigma^{-1}\hat{q}_t + \left(\left(1 - \frac{\varepsilon_M}{\varepsilon_Q}\right)\varepsilon_Q T_1(\mathbf{I} - S_1^M + \varepsilon_M[T_1 - T_2]\right)\hat{p}_t.$$

By definition, we have that:

$$\hat{d}_t^o = \mathbf{W}\left[\hat{m}_t - T_1\hat{\mathbb{M}}_t + T_2\hat{p}_t - T_1\hat{p}_t^{int}\right]_{N^2\times 1}. \tag{121}$$

Substituting in for $\hat{m}_t$, $\hat{\mathbb{M}}_t$, and $\hat{p}_t^{int}$ (which, by definition, equals $S_1^M\hat{p}_t$), we have that:

$$\begin{aligned}\hat{d}_t^o = \mathbf{W}&\left[(\varepsilon_Q - 1)T_1\hat{a}_t + T_1\left[\varepsilon_Q(\Sigma - I) + I\right]\Sigma^{-1}\hat{q}_t + \left(\left(1 - \frac{\varepsilon_M}{\varepsilon_Q}\right)\varepsilon_Q T_1(\mathbf{I} - S_1^M) + \varepsilon_M[T_1 - T_2]\right)\hat{p}_t\right] \\ &- \mathbf{W}\left[(\varepsilon_Q - 1)T_1\hat{a}_t + T_1\left[\varepsilon_Q(\Sigma - I) + I\right]\Sigma^{-1}\hat{q}_t + \varepsilon_Q T_1(\mathbf{I} - S_1^M)\hat{p}_t\right] + \mathbf{W}\left(T_2 - T_1S_1^M\right)\hat{p}_t,\end{aligned}$$

which can be further simplified to yield:

$$\begin{aligned}\hat{d}_t^o &= \mathbf{W}\left[\left(1 - \frac{\varepsilon_M}{\varepsilon_Q}\right)\varepsilon_Q T_1(\mathbf{I} - S_1^M) + \varepsilon_M(T_1 - T_2) - \varepsilon_Q T_1(\mathbf{I} - S_1^M) + \left(T_2 - T_1S_1^M\right)\right]\hat{p}_t \\ &\equiv \mathbf{D}\hat{p}_t.\end{aligned} \tag{122}$$

Therefore, as equation (122) shows, the relationship between outdegrees and prices is independent of the returns to scale. This was also the case in the simple model in section 3.

Finally, recall from equation (108) that we can express the vector of prices, $\hat{p}_t$, as a function of the state variable, $\hat{k}_t$, and the shocks. By substituting for $\hat{p}_t$ into equation (122), we can obtain an equation for the outdegrees as a function of the state variable and

[65] For a derivation of this equation, see page 57 in Atalay (2017).

shocks only, shown as equation (33) in the main text:

$$\begin{aligned}\Delta\hat{d}^o_t &= \mathbf{D}\left[\mathbf{P}_k\hat{k}_t + \mathbf{P}_a\hat{a}_t + \mathbf{P}_d\hat{d}_t\right] \\ &= \mathbf{D}_k\Delta\hat{k}_t + \mathbf{D}_a\omega^A_t + \mathbf{D}_d\omega^D_t. \end{aligned} \tag{123}$$

The above equation assumes away labour-augmenting technology shocks (so $\omega^B_t = 0$ for all t).

C.5 Deriving the log-linearised equation for Domar weights

Since we have taken steady-state labour as the numeraire good (by normalising $\sum_{I=1}^N L_I^{1/\varepsilon_{LS}} = 1$), the aggregate price index (of GDP) cannot be taken as the numeraire. By definition, nominal GDP is given by:

$$NGDP_t = \sum_{J=1}^N P_{tJ}V_{tJ},$$

where V_{tJ} is the value-added of industry J and P_{tJ} the corresponding output price of industry J. In log-linear terms, this is equivalent to:

$$\widehat{NGDP}_t = \sum_{J=1}^N \frac{P_JV_J}{\sum_{J=1}^N P_JV_J}\left(\hat{p}_{tJ} + \hat{v}_{tJ}\right).$$

We can write this as

$$\widehat{NGDP}_t = \mathbf{N}\left(\hat{p}_t + \hat{v}_t\right),$$

where $\mathbf{N}$ is a $1 \times N$ row vector that has $\frac{P_JV_J}{\sum_{J=1}^N P_JV_J}$ in the Jth column.

Now, recall that the Domar weights are given by $\lambda_{tJ} \equiv P_{tJ}Q_{tJ}/NGDP_t$, where $NGDP_t$ denotes nominal GDP at time t. The log-linear approximation to the deviations of Domar weights from their steady-state values is given by:

$$\hat{\lambda}_t = \hat{p}_t + \hat{q}_t - \widehat{NGDP}_t \cdot \iota, \tag{124}$$

where $\widehat{NGDP}_t$ denotes the log-deviation of nominal GDP from its steady state, and ι denotes a vector of ones.

Therefore, we have that $\hat{\lambda}_t = \hat{p}_t + \hat{q}_t - \mathbf{N}'\left(\hat{p}_t + \hat{v}_t\right)$ where $\mathbf{N}'$ is just the row vector $\mathbf{N}$ stacked N times. This can be written as:

$$\hat{\lambda}_t = \left(\mathbf{I} - \mathbf{N}'\right)\hat{p}_t + \hat{q}_t - \mathbf{N}'\hat{v}_t.$$

Using equation (116), we can express the Domar weights as:

$$\hat{\lambda}_t = \underbrace{(\mathbf{I} - \mathbf{N}' + \mathbf{N}'\mathbf{X}_p)}_{\mathbf{W}_p} \hat{p}_t + \underbrace{(\mathbf{I} - \mathbf{N}')}_{\mathbf{W}_q} \hat{q}_t + \underbrace{\mathbf{N}'\mathbf{X}_a}_{\mathbf{W}_a} \hat{a}_t.$$

We want to express the Domar weights in terms of the state variable (capital) and shocks only. We have that:

$$\begin{aligned}
\hat{\lambda}_t &= \mathbf{W}_p \hat{p}_t + \mathbf{W}_q \hat{q}_t + \mathbf{W}_a \hat{a}_t \\
&= \mathbf{W}_p \left[\mathbf{P}_k \hat{k}_t + \mathbf{P}_a \hat{a}_t + \mathbf{P}_b \hat{b}_t + \mathbf{P}_d \hat{d}_t\right] + \mathbf{W}_q \left[\mathbf{\Phi}_k \hat{k}_t + \mathbf{\Phi}_a \hat{a}_t + \mathbf{\Phi}_b \hat{b}_t + \mathbf{\Phi}_d \hat{d}_t\right] + \mathbf{W}_a \hat{a}_t \\
&= \mathbf{\Lambda}_k \hat{k}_t + \mathbf{\Lambda}_a \hat{a}_t + \mathbf{\Lambda}_b \hat{b}_t + \mathbf{\Lambda}_d \hat{d}_t.
\end{aligned}$$

Taking differences on both sides and assuming away labour-augmenting technology shocks, we obtain the bottom row of equation (32) shown in the main text:

$$\Delta \hat{\lambda}_t = \mathbf{\Lambda}_k \Delta \hat{k}_t + \mathbf{\Lambda}_a \omega_t^A + \mathbf{\Lambda}_d \omega_t^D. \tag{125}$$

C.6 Deriving the log-linearised equation for labour

From the bottom of page 60 in Atalay (2017), we have that:

$$\hat{l}_t = \mathbf{\Lambda}_k \hat{k}_t + \mathbf{\Lambda}_a \hat{a}_t + \mathbf{\Lambda}_b \hat{b}_t + \mathbf{\Lambda}_p \hat{p}_t.$$

Substituting in for $\hat{p}_t$ using equation (108), we have that:

$$\begin{aligned}
\hat{l}_t &= \mathbf{\Lambda}_k \hat{k}_t + \mathbf{\Lambda}_a \hat{a}_t + \mathbf{\Lambda}_b \hat{b}_t + \mathbf{\Lambda}_p \left[\mathbf{P}_k \hat{k}_t + \mathbf{P}_a \hat{a}_t + \mathbf{P}_b \hat{b}_t + \mathbf{P}_d \hat{d}_t\right] \\
&= \mathbf{L}_k \hat{k}_t + \mathbf{L}_a \hat{a}_t + \mathbf{L}_b \hat{b}_t + \mathbf{L}_d \hat{d}_t. \qquad (126)
\end{aligned}$$

One period ahead:

$$\begin{aligned}
\hat{l}_{t+1} &= \mathbf{L}_k \hat{k}_{t+1} + \mathbf{L}_a \hat{a}_{t+1} + \mathbf{L}_b \hat{b}_{t+1} + \mathbf{L}_d \hat{d}_{t+1} \\
&= \mathbf{L}_k \left(M_{kk} \hat{k}_t + M_{ka} \hat{a}_t + M_{kb} \hat{b}_t + M_{kd} \hat{d}_t\right) + \mathbf{L}_a \hat{a}_{t+1} + \mathbf{L}_b \hat{b}_{t+1} + \mathbf{L}_d \hat{d}_{t+1}
\end{aligned}$$

where we make use of equation (106) in the second line above. As long as $\mathbf{L}_k$ is invertible, we can solve for $\hat{k}_t$ using equation (126) and substitute $\hat{k}_t$ from the above equation to obtain:

$$\begin{aligned}
\hat{l}_{t+1} = \mathbf{L}_k \Big(M_{kk} \left[\mathbf{L}_k^{-1} \hat{l}_t - \mathbf{L}_k^{-1} \mathbf{L}_a \hat{a}_t - \mathbf{L}_k^{-1} \mathbf{L}_b \hat{b}_t - \mathbf{L}_k^{-1} \mathbf{L}_d \hat{d}_t\right] &+ M_{ka} \hat{a}_t + M_{kb} \hat{b}_t + M_{kd} \hat{d}_t\Big) \\
&+ \mathbf{L}_b \hat{b}_{t+1} + \mathbf{L}_a \hat{a}_{t+1} + \mathbf{L}_d \hat{d}_{t+1},
\end{aligned}$$

so we finally get:

$$\Delta\hat{l}_{t+1} = \tilde{\mathbf{L}}_l \Delta\hat{l}_t + \mathbf{L}_a \omega_{t+1}^A + \mathbf{L}_b \omega_{t+1}^B + \mathbf{L}_d \omega_{t+1}^D + \tilde{\mathbf{L}}_a \omega_t^A + \tilde{\mathbf{L}}_b \omega_t^B + \tilde{\mathbf{L}}_d \omega_t^D. \tag{127}$$

By assuming away labour-augmenting technology shocks, we recover the bottom row of equation (42) shown in the main text.

D. Robustness Analysis for Section 4

Table D.1: Implied Size Centrality Relationship in Dynamic Model Under Different Values of ε_Q and ε_D vs. Empirical Data Counterpart for the UK

Implied Size-Centrality Relationship	Value of ε_Q			Value of ε_D			Data
	0.5	1 (*baseline*)	1.3	0.8	1 (*baseline*)	1.3	
$\overline{\text{corr}}(\hat{d}_t^{out}, \hat{q}_t)$ given all shocks	-0.18	0.00	0.19	-0.05	0.00	-0.02	0.12
$\overline{\text{corr}}(\hat{d}_t^{out}, \hat{\lambda}_t)$ given all shocks	0.06	0.23	0.41	0.21	0.23	0.25	0.17
$\overline{\text{corr}}(\hat{d}_t^{out}, \hat{q}_t)$ given only technology shocks	-0.78	-0.67	-0.52	-0.60	-0.67	-0.71	n.a.
$\overline{\text{corr}}(\hat{d}_t^{out}, \hat{\lambda}_t)$ given only technology shocks	0.61	0.58	0.49	0.76	0.58	0.19	n.a.
$\overline{\text{corr}}(\hat{d}_t^{out}, \hat{q}_t)$ given only preference shocks	0.96	0.86	0.84	0.87	0.86	0.83	n.a.
$\overline{\text{corr}}(\hat{d}_t^{out}, \hat{\lambda}_t)$ given only preference shocks	0.97	0.90	0.90	0.91	0.90	0.90	n.a.

Notes: $\hat{d}_t^{out}$ denotes producer centrality, $\hat{q}_t$ denotes real gross output, and $\hat{\lambda}_t$ denotes Domar weight (all in terms of steady-state log deviations).

Table D.2: Contributions of Sector-Specific/Common Shocks to UK's Post-2010 Productivity Growth Puzzle for Different Values of ε_Q and ε_D

Contribution to Productivity Growth Puzzle	Value of ε_Q			Value of ε_D			Data
	0.5	1 (*baseline*)	1.3	0.8	1 (*baseline*)	1.3	
Manufacturing-specific shocks	-0.54	-0.65	-0.66	-0.65	-0.65	-0.66	n.a.
Finance-specific shocks	-0.00	-0.04	-0.09	-0.02	-0.04	-0.06	n.a.
Other sectors' specific shocks	0.15	0.30	0.40	0.35	0.30	0.25	n.a.
Common shocks	0.15	0.13	0.09	0.16	0.13	0.12	n.a.
Total growth puzzle	-0.24	-0.26	-0.26	-0.15	-0.26	-0.35	-0.18
Correlation of model-implied aggregate productivity growth and data counterpart	0.76	0.77	0.74	0.75	0.77	0.76	n.a.

Notes: All other model parameters set according to the baseline calibration, shown in Table 3.

Figure D.1. Contributions to the growth puzzle: sectors vs. sectoral shocks (dashed)

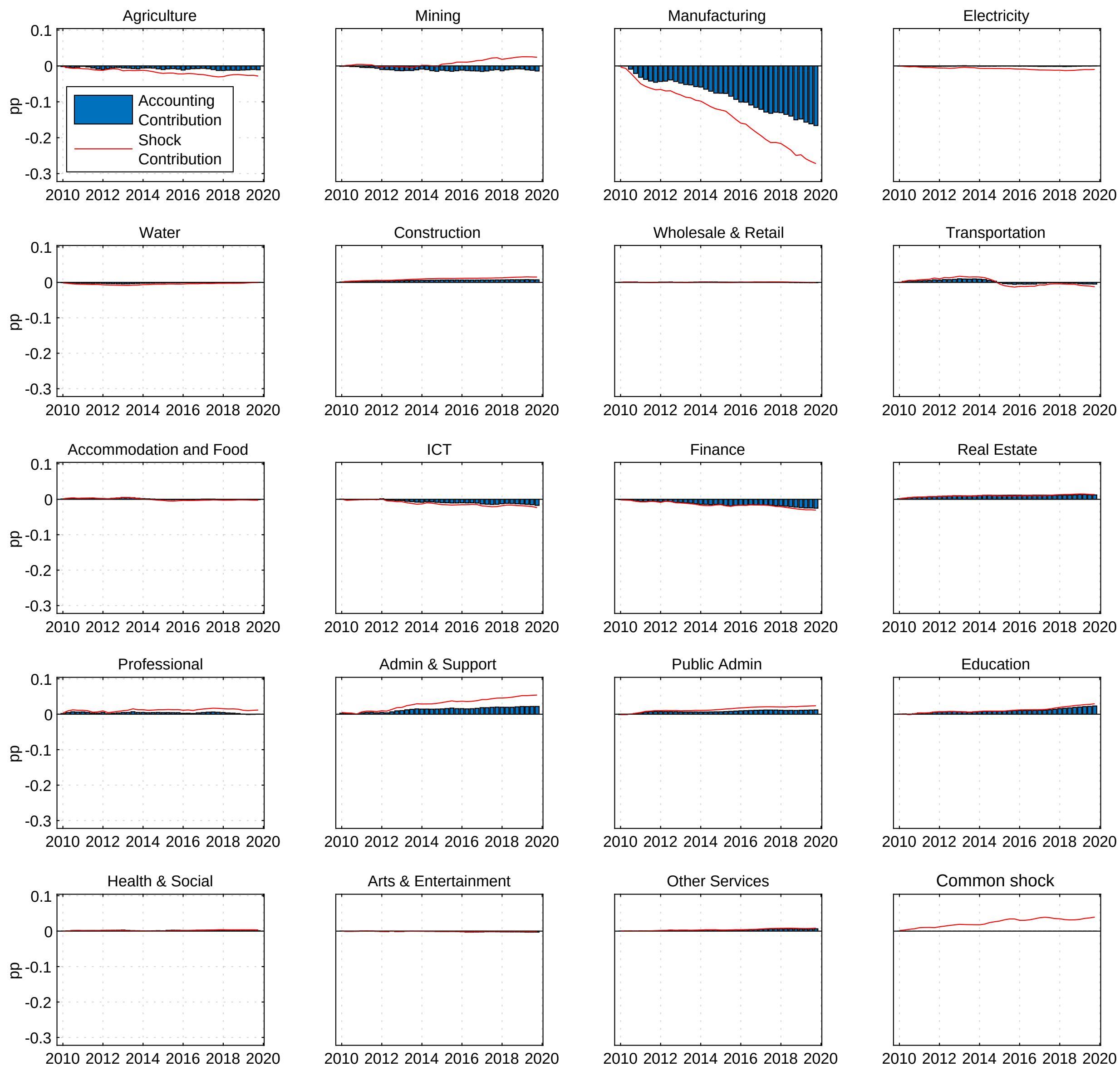


Figure D.2. Robustness of Baseline Results in Figure 14 in Alternative Parametrisations